Reducing hopeful majority

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We are given a non-empty bag of (votes on) 'candidates', and are asked to determine if some candidate has the majority.

Several derivations of linear-time algorithms have been given, all of which work in two phases: first find a 'hopeful majority' candidate, and next check if it really has the majority. A 'hopeful majority' candidate is any candidate \( c \) in the bag such that if some candidate has the majority, it is \( c \).

I consider here only the problem of finding some hopeful majority candidate. All previous algorithms I have seen basically scan the bag. The purpose of this note is to show that there is a divide-and-rule approach. In a previous note I have given a derivation, mainly based on predicate calculus. Here only the solution is presented.

Let \( C \) stand for the type of the candidates, and \( N \) for the naturals. The operation \( \oplus: (C \times N) \times (C \times N) \rightarrow C \times N \) is defined by:

\[
(c0, d0) \oplus (c1, d1) = (c0, d0 + d1) \; \langle c0 = c1 > ((c0, d0 - d1) \uparrow \pi_2 (c1, d1 - d0)) ,
\]

where \( a < p > b \) stands for if \( p \) then \( a \) else \( b \). We also define \( f: C \rightarrow C \times N: \)

\[
f \cdot c = (c, 1) .
\]

Now a hopeful majority candidate is determined by \( \pi_2 \cdot h \), where \( h \) is the bag homomorphism defined by

\[
h = \oplus / \cdot f^* .
\]

However, something funny is going on here. Even under the hot indeterminate interpretation of \( \uparrow \pi_2 \) the operator \( \oplus \) is not associative. This can be seen by considering the different ways to compute \( h \) on a bag of three distinct candidates. According to the current definitions this would mean that \( h \) is not a proper bag homomorphism. Associativity is not required for consistency if we consider the bag splitting itself also as indeterminate. This is already mentioned in the *Algorithmics* paper, but I had not come across a clear (non-contrived) example before. The definition of the indeterminate bag reduce \( \oplus / \) is then that it is the 'thinnest' (most determinate) indeterminate function \( r \) satisfying

\[
r \cdot \tau \leadsto c .
\]