The Dimension of the Vector Space Spanned by Sets of Radio-Interferometric Measurements*

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Abstract

A novel technique developed at Vanderbilt University, based on radio interferometry, makes it possible to determine inter-node distances in 2D or 3D ad-hoc wireless networks with far greater accuracy than was possible with earlier methods. In this note we determine an upper bound on the number of independent measurements on a set of n nodes, and prove it to be sharp.

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1 Introduction

An outstanding problem for casually deployed wireless sensor networks is the development of methods that allow the nodes in the network to determine their positions relative to each other — or absolutely if the network is seeded with anchor nodes, which have *a priori* knowledge of their absolute positions.

Given accurate estimates of the distances between nearby nodes, it is possible to construct accurate position maps in a distributed fashion [2].

A novel technique developed by Maróti *et al.* [1], based on radio interferometry, makes it possible to determine inter-node distances in 2D or 3D ad-hoc wireless networks with far greater accuracy than was possible with earlier methods. However, instead of directly yielding quantities d_{AB} for the distances between a pair of nodes A and B, the technique measures quantities $d_{AC} - d_{AD} - d_{BC} + d_{BD}$ for quadruples of distinct nodes (A, B, C, D).

These measurements are not all independent. In this note we determine an upper bound on the number of independent measurements on a set of n nodes, and prove it to be sharp. This sharpens the upper bound given in [1], Theorem 4, while also repairing a problem with its proof, in which measurements may be reduced to measurements that cannot actually be made.

2 Statement of the problem

We assume that the network has at least three nodes, and that the nodes forming the network are numbered 0 through $n{-}1$, where n is at least 3. Let $N = \{0 \mathinner{..} n{-}1\}$ denote the set of nodes. In the notation d_{AB} we always assume that A and B are nodes in N. By convention, d_{BA} means the same as d_{AB} — a fact that would be obvious if the notation had been $d_{\{A,B\}}$. Clearly, there is no need to determine quantities d_{AA}, so without loss we require in the notation d_{AB} that $A \neq B$. Then in the network there are in all $n(n{-}1)/2$ such quantities d_{AB}.

These distances are not independent in the sense of being mutually unconstrained. To start with, there is the triangle inequality: $d_{AC} \leq d_{AB} + d_{BC}$. Assuming that the nodes live in 2D space, there is further the constraint that the Cayley-Menger determinant on any quadruple of nodes (A, B, C, D) vanishes [3], where that determinant is given by:

$$\begin{vmatrix} 0 & d^2_{AB} & d^2_{AC} & d^2_{AD} & 1 \\ d^2_{AB} & 0 & d^2_{BC} & d^2_{BD} & 1 \\ d^2_{AC} & d^2_{BC} & 0 & d^2_{CD} & 1 \\ d^2_{AD} & d^2_{BD} & d^2_{CD} & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} .$$

Here we are concerned with a more technical notion of independence: linear independence of a collection of vectors in a vector space.

Recall that, given a vector space V and a set of vectors $\{v_i\}_i$ of V, the subspace *spanned* by that set consists of the collection of vectors that can be written as $\sum_i \lambda_i v_i$ for some assignment of scalar values λ_i. The set of vectors is called *linearly independent* when $\sum_i \lambda_i v_i = 0 \quad \equiv \quad \forall_i\ \lambda_i = 0$. A *basis* of V is then a linearly independent set of vectors of V that spans V.

Now take an $n(n-1)/2$ dimensional vector space over the field of the real numbers, and label the vectors of some basis with d_{AB}, for A and B distinct nodes from N — also here label d_{AB} is identified with label d_{BA}.

Define, for (A, B, C, D) a quadruple of distinct nodes,

$$d_{ABCD} = d_{AC} - d_{AD} - d_{BC} + d_{BD} \ .$$

So d_{ABCD} is a vector in our vector space. We call it a *measurement*, because it corresponds to a possible measurement that could be carried out by the new radio-interferometric technique, the outcome being (modulo experimental error) the value of the right-hand side under some valuation of the basis vectors d_{AB}. Clearly, $d_{ABCD} + d_{BACD} = 0$ and $d_{ABCD} - d_{CDAB} = 0$, so these vectors are not all mutually independent. To rule out these pairwise dependencies, we require that in any index $ABCD$ we have:

$$A < B, \quad A < C < D, \quad B \neq C, \quad B \neq D \ ,$$

in which the last two inequalities, required by the distinctness of the four nodes, are given for the sake of completeness. We call an index satisfying these inequalities *normalized*. If some of the other inequalities are violated, the corresponding measurement can be found from one with a normalized index by using the pairwise dependencies given above.

Since there are three orderings of A, B, C and D compatible with the index inequalities, *viz.* $A{<}B{<}C{<}D$, $A{<}C{<}B{<}D$ and $A{<}C{<}D{<}B$, any choice of four distinct nodes from

N leads to three normalized indices, and so the set of normalized indices $ABCD$ has size $3\binom{n}{4} = n(n-1)(n-2)(n-3)/8$.

We want to determine the dimension of the vector space spanned by the set of all possible measurements; *i.e.*, the size of a basis of that space. This is then the size of any maximally large set of linearly independent measurements.

3 The dimension

THEOREM. *The dimension of the vector space spanned by the measurements d_{ABCD} on a set of n nodes, $n \geq 3$, is $n(n-3)/2$.*

PROOF. Partition the set of normalized indices into six classes:

Class 0: $\{\, 0\,1\,2\,D \mid 2 < D\}$
containing $n-3$ elements;

Class 1: $\{\, 0\,B\,1\,D \mid 1 < B < D\}$
containing $\binom{n-2}{2} = (n-2)(n-3)/2$ elements;

Class 2: $\{\, 0\,1\,CD \mid 2 < C < D\}$
containing $\binom{n-3}{2} = (n-3)(n-4)/2$ elements;

Class 3: $\{\, 0\,B\,1\,D \mid 1 < D < B\}$
containing $\binom{n-2}{2} = (n-2)(n-3)/2$ elements;

Class 4: $\{\, 0\,BCD \mid 1 < B, 1 < C < D, B \neq C, B \neq D\}$
containing $3\binom{n-2}{3} = (n-2)(n-3)(n-4)/2$ elements;

Class 5: $\{ABCD \mid 0 < A < B, A < C < D, B \neq C, B \neq D\}$
containing $3\binom{n-1}{4} = (n-1)(n-2)(n-3)(n-4)/8$ elements.

It is easily verified that all these indices are normalized and that the classes are disjoint. The sizes sum up to the cardinality of the set of all normalized indices, so this indeed constitutes a partitioning.

First we show that the measurements having indices of classes 0 and 1 together form a linearly independent set. Next, we show that all measurements indexed by elements of classes 2–5 can be reduced to a linear combination of measurements with lower class numbers, ultimately leading to a linear combination of elements from classes 0 and 1. Combined, this gives us that classes 0 and 1 together form a basis. Since there are $n(n-3)/2$ elements in these two classes, the claim then follows.

As to the linear independence of the measurements indexed by classes 0 and 1, assume some linear combination of these measurements vanishes:

$$\sum_{2<D} \lambda_D d_{012D} + \sum_{1<B<D} \mu_{BD} d_{0B1D} \; = \; 0 \, ,$$

or, equivalently, using the definition of d_{ABCD}:

$$\sum_{2<D} \lambda_D (d_{02} - d_{0D} - d_{12} + d_{1D}) + \sum_{1<B<D} \mu_{BD} (d_{01} - d_{0D} - d_{1B} + d_{BD}) \; = \; 0 \, .$$

Recall that the vectors d_{AB} form a basis. The coefficient of each vector d_{BD}, $1<B<D$, is μ_{BD}. So each $\mu_{BD} = 0$. Then the coefficient of each vector d_{1D}, $2<D$, is λ_D. So also each $\lambda_D = 0$. Therefore a linear combination of the measurements only vanishes if all coefficients are zero: they are independent.

The reductions of measurements from higher classes to classes 0 and 1 are as follows:

Class 2: $d_{01CD} = -d_{012C} + d_{012D}$;
Class 3: $d_{0B1D} = -d_{01DB} + d_{0D1B}$;
Class 4: $d_{0BCD} = -d_{0B1C} + d_{0B1D}$;
Class 5: $d_{ABCD} = -d_{0ACD} + d_{0BCD}$.

In each case it is straightforward to verify that the measurements in the right-hand side are indexed by indices from a lower class. For example, the index $0D1B$ occurring in the reduction for class 3 has $D < B$ (because the left-hand side satisfies the constraints of class 3) and therefore belongs to class 1. It is equally straightforward to verify that each reduction represents a valid identity; for example, for class 3, expanding the definition of d_{ABCD} and using $d_{BA} = d_{AB}$, we obtain:

$$d_{01} - d_{0D} - d_{1B} + d_{BD} \;=\; -\,(d_{0D} - d_{0B} - d_{1D} + d_{1B}) + (d_{01} - d_{0B} - d_{1D} + d_{BD}) \, ,$$

which is easily seen to hold.

□

4 Conclusion

For $n > 4$ the upper bound $n(n-3)/2$ proved here improves on the upper bound $(n-2)(n-3)$ given in [1], and is sharp. Also, our proof overcomes the deficiency in the proof given there. For a 6-node network this means there are at most 9 independent measurements, just enough for the 9 positioning unknowns in the 2D case, but falling short of the 12 unknowns for the 3D case. There, at least an 8-node network is needed, 8 being the least value of n for which $n(n-3)/2 \geq 3n-6$, the number of unknowns for a 3D n-node network.

References

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