Mechanizing the Development of Software

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Abstract. This paper presents a mechanizable framework for software development by refinement. The framework is based on a category of specifications. The key idea is representing knowledge about programming concepts, such as algorithm design, datatype refinement, and expression simplification, by means of taxonomies of specifications and morphisms. Examples are drawn from working systems Specware, Designware, and Planware.
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1. Overview

A software system can be viewed as a composition of information from a variety of sources, including

- the application domain,
- the requirements on the system’s behavior,
- software design knowledge about system architectures, algorithms, data structures, code optimization techniques, and
- the run-time hardware/software/physical environment in which the software will execute.

This paper presents a mechanizable framework for representing these various sources of information, and for composing them in the context of a refinement process. The framework is founded on a category of specifications. Morphisms are used to structure and parameterize specifications, and to refine them. Colimits are used to compose specifications. Diagrams are used to express the structure of large specifications, the refinement of specifications to code, and the application of design knowledge to a specification.

The framework features a collection of techniques for constructing refinements based on formal representations of programming knowledge. Abstract algorithmic concepts, datatype refinements, program optimization rules, software architectures, abstract user interfaces, and so on, are represented as diagrams of specifications and morphisms. We arrange these diagrams into taxonomies, which allow incremental access to and construction of refinements for particular requirement specifications. For example, a user may specify a scheduling problem and select a theory of global search algorithms from an algorithm library. The global search theory is used to construct a refinement of the scheduling problem specification into a specification containing a global search algorithm for the particular scheduling problem.

The framework is partially implemented in the research systems Specware, Designware, and Planware. Specware provides basic support for composing specifications and refinements, and generating code. Code generation in Specware is supported by interlogic morphisms that translate between the specification language/logic and the logic of a particular programming language (e.g. CommonLisp or C++). Specware is intended to be general-purpose and has found use in industrial settings. Designware extends Specware with taxonomies of software design theories and support for constructing refinements from them. Planware provides highly automated support for requirements acquisition and synthesis of high-performance scheduling algorithms.

The remainder of this paper covers basic concepts and the key ideas of our approach to software development by refinement, in particular the concept of design by classification [9]. A simple example is carried through algorithm design, datatype refinement, and expression simplification. We also discuss the application of these techniques to domain-specific refinement in Planware [1].

This paper assumes a rudimentary knowledge of logic and category theory, although details about the categories of interest are introduced and motivated by examples.
spec Container is
  sorts E, Container
  op empty :→ Container
  op singleton : E → Container
  op _join_ : Container, Container → Container
  constructors {empty, singleton, join} construct Container

  axiom \( \forall (x : \text{Container})(x \text{ join empty} = x \land \text{empty join} x = x) \)
  op _in_ : E, Container → Boolean
  definition of in is
    axiom \(x \text{ in empty} = \text{false}\)
    axiom \(x \text{ in singleton}(y) = (x = y)\)
    axiom \(x \text{ in } U \text{ join } V = (x \text{ in } U \lor x \text{ in } V)\)
  end-definition

end-spec

Figure 1: Specification for Containers

2. Basic Concepts

2.1. Specifications

A specification is the finite presentation of a theory. The signature of a specification provides the vocabulary for describing objects, operations, and properties in some domain of interest, and the axioms constrain the meaning of the symbols. The theory of the domain is the closure of the axioms under the rules of inference.

Example: Here is a specification for partial orders, using notation adapted from Specware. It introduces a sort \(E\) and an infix binary predicate on \(E\), called \(\le\), which is constrained by the usual axioms. Although Specware allows higher-order specifications, first-order formulations are sufficient in this paper.

spec Partial-Order is
  sort E
  op \(\le\) : E, E → Boolean
  axiom reflexivity is \(x \le x\)
  axiom transitivity is \(x \le y \land y \le z \implies x \le z\)
  axiom antisymmetry is \(x \le y \land y \le x \implies x = z\)
end-spec

Example: Containers are constructed by a binary join operator and they represent finite collections of elements of some sort \(E\). The specification shown in Figure 1 includes a definition by means of axioms. Operators are required to be total. The constructor clause asserts that the operators \{empty, singleton, join\} construct the sort \(\text{Container}\), providing the basis for induction on \(\text{Container}\).

The generic term expression will be used to refer to a term, formula, or sentence.
A model of a specification is a structure of sets and total functions that satisfy the
axioms. However, for software development purposes we have a less well-defined notion of
semantics in mind: each specification denotes a set of possible implementations in some
computational model. Currently we regard these as functional programs. A denotational
semantics maps these into classical models.

2.2. Morphisms
A specification morphism translates the language of one specification into the language
of another specification, preserving the property of provability, so that any theorem in
the source specification remains a theorem under translation.

A specification morphism \( m : T \rightarrow T' \) is given by a map from the sort and operator
symbols of the domain \( T \) to the symbols of the codomain \( T' \). To be a specification morphism it is also required that every axiom of \( T \) translates to a theorem
of \( T' \). It then follows that a specification morphism translates theorems of the domain
specification to theorems of the codomain.

Example: A specification morphism from \( \text{Partial-Order} \) to \( \text{Integer} \) is:

\[
\text{morphism } \text{Partial-Order-to-Integer} \text{ is }
\{ E \mapsto \text{Integer}, \text{le} \mapsto \leq \}
\]

Translation of an expression by a morphism is by straightforward application of the
symbol map, so, for example, the \( \text{Partial-Order} \) axiom \( x \text{ le} x \) translates to \( x \leq x \). The
three axioms of a partial order remain provable in \( \text{Integer} \) theory after translation.

Example: A parameterized specification can be treated as a morphism. The specification \( \text{Bag} \) in Appendix A can be parameterized on a spec \( \text{Triv} \) with a single sort:

\[
\text{spec } \text{Triv} \text{ is }
\text{sort } E
\text{end-spec}
\]

via the morphism

\[
\text{morphism } \text{Bag-Parameterization} \text{ is }
\{ E \mapsto E \}
\]

The semantics of a specification morphism is given by a contravariant functor \( \text{Mod} \)
that maps each specification to a category of models and each specification morphism
\( m : T \rightarrow T' \) to a reduct functor \( \cdot |_m : \text{Mod}(T') \rightarrow \text{Mod}(T) \).

Morphisms come in a variety of flavors; here we only use two. An extension or import
is an inclusion between specs.

Example: We can build up the theory of partial orders by importing the theory of
preorders. The import morphism is \( \{ E \mapsto E, \text{le} \mapsto \le \} \).

\[
\text{spec } \text{PreOrder} \text{ is }
\text{sort } E
\text{op } \leq \text{: } E, E \rightarrow \text{Boolean}
\]
axiom reflexivity is $x \leq x$
axiom transitivity is $x \leq y \land y \leq z \implies x \leq z$
\end-spec

spec Partial-Order
  import PreOrder
  axiom antisymmetry is $x \leq y \land y \leq x \implies x = z$
\end-spec

A definition extension, written $A \rightarrow B$, is an import morphism in which any new symbol in $B$ also has an axiom that defines it. Definitions have implicit axioms for existence and uniqueness. Semantically, a definition extension has the property that each model of the domain has a unique expansion to a model of the codomain.

Example: Container can be formulated as a definition extension of Pre-Container:

spec Pre-Container is
  sorts $E$, Container
  op empty : $\rightarrow$ Container
  op singleton : $E \rightarrow$ Container
  op join : Container, Container $\rightarrow$ Container
  constructors \{empty, singleton, join\} construct Container
  axiom $\forall (x : \text{Container})(x \text{ join } \text{empty} = x \land \text{empty join } x = x)$
\end-spec

spec Container is
  imports Pre-Container
  definition of in is
    axiom $x$ in empty = false
    axiom $x$ in singleton$(y) = (x = y)$
    axiom $x$ in $U$ join $V = (x$ in $U \lor x$ in $V)$
  end-definition
\end-spec

2.3. The Category of Specs

Specification morphisms compose in a straightforward way as the composition of finite maps. It is easily checked that specifications and specification morphisms form a category SPEC. Colimits exist in SPEC and are easily computed. Suppose that we want to compute the colimit of $B \xrightarrow{i} A \xrightarrow{j} C$. First, form the disjoint union of all sort and operator symbols of $A$, $B$, and $C$, then define an equivalence relation on those symbols:

$s \approx t$ iff $(i(s) = t \lor i(t) = s \lor j(s) = t \lor j(t) = s)$.

The signature of the colimit (also known as pushout in this case) is the collection of equivalence classes wrt $\approx$. The cocone morphisms take each symbol into its equivalence
class. The axioms of the colimit are obtained by translating and collecting each axiom of \(A, B,\) and \(C.\)

**Example:** Suppose that we want to build up the theory of partial orders by composing simpler theories.

\[
\begin{array}{c}
\text{spec } \text{BinRel is} \\
\text{sort } E \\
\text{op } \_le_\ : \ E, E \to \text{Boolean} \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{spec } \text{PreOrder is} \\
\text{import } \text{BinRel} \\
\text{axiom } \text{reflexivity is } x \_le_\ x \\
\text{axiom } \text{transitivity is} \\
x \_le_\ y \land y \_le_\ z \implies x \_le_\ z \\
\end{array}
\]

\[
\begin{array}{c}
\text{spec } \text{Antisymmetry is} \\
\text{import } \text{BinRel} \\
\text{axiom } \text{antisymmetry is} \\
x \_le_\ y \land y \_le_\ x \implies x = z \\
\end{array}
\]

The pushout of \(\text{Antisymmetry} \leftarrow \text{BinRel} \to \text{PreOrder}\) is isomorphic to the specification for \(\text{Partial-Order}\) in Section 2.1. In detail: the morphisms are \(\{E \mapsto E, \_le_\mapsto \_le_\}\) from \(\text{BinRel}\) to both \(\text{PreOrder}\) and \(\text{Antisymmetry}\). The equivalence classes are then \(\{E, E, E\}, \{\_le_\, \_le_\, \_le_\}\), so the colimit spec has one sort (which we rename \(E\)), and one operator (which we rename \(\_le_\)). Furthermore, the axioms of \(\text{BinRel, Antisymmetry,}\) and \(\text{PreOrder}\) are each translated to become the axioms of the colimit. Thus we have \(\text{Partial-Order}\).

In the category of specifications, colimit acts as a kind of union operator: the pushout collects the symbols of \(\text{Antisymmetry}\) and \(\text{PreOrder}\) where the morphisms from \(\text{BinRel}\) indicate which symbols are shared. The colimit has the universal property that it computes the simplest such specification.

**Example:** The pushout operation is also used to instantiate the parameter in a parameterized specification. The binding of argument to parameter is represented by a morphism. To form a specification for bags of integers, we compute the pushout of \(\text{Bag} \leftarrow \text{Triv} \to \text{Integer}\), where \(\text{Bag} \leftarrow \text{Triv}\) is \(\{E \mapsto E\}\), and \(\text{Triv} \to \text{Integer}\) is \(\{E \mapsto \text{Integer}\}\).

**Example:** A specification for sequences can be built up from \(\text{Container}\), also via pushouts. We can regard \(\text{Container}\) as parameterized on a binary operator

\[
\begin{array}{c}
\text{spec } \text{BinOp is} \\
\text{sort } E \\
\text{op } \_bop_\ : \ E, E \to E \\
\end{array}
\]

morphism \(\text{Container-Parameterization : BinOp } \to \text{Container}\) is \(\{E \mapsto E, \_bop_\mapsto \text{join}\}\) and we can define a refinement arrow that extends a binary operator to a semigroup:
spec Associativity is
import BinOp
axiom Associativity is \((x \text{ join } y) \text{ join } z = (x \text{ join } (y \text{ join } z))\)
end-spec

The pushout of \(\text{Associativity} \leftarrow \text{BinOp} \rightarrow \text{Container}\), produces a collection specification with an associative join operator, which is \(\text{Proto-Seq}\), the core of sequence theory in Appendix A.

### 2.4. Diagrams

Roughly, a *diagram* is a graph morphism to a category, usually the category of specifications in this paper. For example, the pushout described above started with a diagram comprised of two arrows:

\[
\begin{align*}
\text{BinRel} & \quad \longrightarrow \quad \text{PreOrder} \\
\downarrow & \quad \downarrow \\
\text{Antisymmetry} & \\
\end{align*}
\]

and computing the pushout of that diagram produces another diagram:

\[
\begin{align*}
\text{BinRel} & \quad \longrightarrow \quad \text{PreOrder} \\
\downarrow & \quad \downarrow \\
\text{Antisymmetry} & \quad \longrightarrow \quad \text{Partial-Order} \\
\end{align*}
\]

A diagram *commutes* if the composition of arrows along two paths with the same start and finish node yields equal arrows.

#### 2.4.1. The Structuring of Specifications

Colimits can be used to construct a large specification from a diagram of specs and morphisms. The morphisms express various relationships between specifications, including sharing of structure, inclusion of structure, and parametric structure. Several examples will appear later.

*Example:* The finest-grain way to compose \(\text{Partial-Order}\) is via the colimit of

\[
\begin{align*}
\text{BinRel} & \quad \longrightarrow \quad \text{Partial-Order} \\
\downarrow & \quad \downarrow \\
\text{Reflexivity} & \quad \text{Transitivity} & \quad \text{Antisymmetry} \\
\end{align*}
\]

*Example:* Appendix B gives a structured specification for sorting bags over a linear order.
2.4.2. Refinement and Diagrams

As described above, specification morphisms can be used to help structure a specification, but they can also be used to refine a specification. When a morphism is used as a refinement, the intended effect is to reduce the number of possible implementations when passing from the domain spec to the codomain. In this sense, a refinement can be viewed as embodying a particular design decision or property that corresponds to the subset of possible implementations of the domain spec which are also possible implementations of the codomain.

Often in software refinement we want to preserve and extend the structure of a structured specification (versus flattening it out via colimit). When a specification is structured as a diagram, then the corresponding notion of structured refinement is a diagram morphism. A diagram morphism $M$ from diagram $D$ to diagram $E$ consists of a set of specification morphisms, one from each node/spec in $D$ to a node in $E$ such that certain squares commute (a functor underlies each diagram and a natural transformation underlies each diagram morphism). We use the notation $D \rightarrow E$ for diagram morphisms.

Example: A datatype refinement that refines bags to sequences can be presented as the diagram morphism $BtoS : BAG \rightarrow BAG-AS-SEQ$:

![Diagram morphism](image)

where the domain and codomain of $BtoS$ are shown in boxes, and the (one) square commutes. Here $Bag-as-Seq$ is a definitional extension of $Seq$ that provides an image for $Bag$ theory. Specs for $Bag$, $Seq$ and $Bag-as-Seq$ and details of the refinement can be found in Appendix A. The interesting content is in spec morphism $BtoS_{Bag}$:

morphism $BtoS_{Bag} : Bag \rightarrow Bag-as-Seq$ is

\[
\begin{align*}
{Bag} & \mapsto \text{Bag-as-Seq}, \\
{empty-bag} & \mapsto \text{bag-empty}, \\
{empty-bag?} & \mapsto \text{bag-empty?}, \\
{nonempty?} & \mapsto \text{bag-nonempty?}, \\
{singleton-bag} & \mapsto \text{bag-singleton}, \\
{singleton-bag?} & \mapsto \text{bag-singleton?}, \\
{nonsingleton-bag?} & \mapsto \text{bag-nonsingleton?}, \\
{in} & \mapsto \text{bag-in}, \\
{bag-union} & \mapsto \text{bag-union}, \\
{bag-wfgt} & \mapsto \text{bag-wfgt}, \\
{size} & \mapsto \text{bag-size}
\end{align*}
\]

Diagram morphisms compose in a straightforward way based on spec morphism composition. It is easily checked that diagrams and diagram morphisms form a category. Colimits in this category can be computed using colimits in SPEC. In the sequel we will generally use the term refinement to mean a diagram morphism.
2.5. Logic Morphisms and Code Generation

Inter-logic morphisms [2] are used to translate specifications from the specification logic to the logic of a programming language. See [10] for more details. They are also useful for translating between the specification logic and the logic supported by various theorem-provers and analysis tools. They are also useful for translating between the theory libraries of various systems.

3. Software Development by Refinement

The development of correct-by-construction code via a formal refinement process is shown to the left. The refinement process starts with a specification $S_0$ of the requirements on a desired software artifact. Each $S_i$, $i = 0, 1, ..., n$ represents a structured specification (diagram) and the arrows $\Downarrow$ are refinements (represented as diagram morphisms). The refinement from $S_i$ to $S_{i+1}$ embodies a design decision which cuts down the number of possible implementations. Finally an inter-logic morphism translates a low-level specification $S_n$ to code in a programming language. Semantically the effect is to narrow down the set of possible implementations of $S_n$ to just one, so specification refinement can be viewed as a constructive process for proving the existence of an implementation of specification $S_0$ (and proving its consistency).

Clearly, two key issues in supporting software development by refinement are: (1) how to construct specifications, and (2) how to construct refinements. Most of the sequel treats mechanizable techniques for constructing refinements.

3.1. Constructing Specifications

A specification-based development environment supplies tools for creating new specifications and morphisms, for structuring specs into diagrams, and for composing specifications via importation, parameterization, and colimit. In addition, a software development environment needs to support a large library of reusable specifications, typically including specs for (1) common datatypes, such as integer, sequences, finite sets, etc. and (2) common mathematical structures, such as partial orders, monoids, vector spaces, etc. In addition to these generic operations and libraries, the system may support specialized construction tools and libraries of domain-specific theories, such as resource theories, or generic theories about domains such as satellite control or transportation.
3.2. Constructing Refinements

A refinement-based development environment supplies tools for creating new refinements. One of our innovations is showing how a library of abstract/reusable/generic refinements can be applied to produce refinements for a given specification. In this paper we focus mainly on refinements that embody design knowledge about (1) algorithm design, (2) datatype refinement, and (3) expression optimization. We believe that other types of design knowledge can be similarly expressed and exploited, including interface design, software architectures, domain-specific requirements capture (see Section 4. on Planware), and others. In addition to these generic operations and libraries, the system may support specialized construction tools and libraries of domain-specific refinements.

The key concept of this work is the following: abstract design knowledge about datatype refinement, algorithm design, software architectures, program optimization rules, visualization displays, and so on, can be expressed as refinements (i.e. diagram morphisms). The domain of one such refinement represents the abstract structure that is required in a user’s specification in order to apply the embodied design knowledge. The refinement itself embodies a design constraint – the effect is a reduction in the set of possible implementations. The codomain of the refinement contains new structures and definitions that are composed with the user’s requirement specification.

\[
\begin{array}{ccc}
A & \rightarrow & S_0 \\
\downarrow & & \downarrow \\
B & \rightarrow & S_1
\end{array}
\]

The figure to the left shows the application of a library refinement \(A \Rightarrow B\) to a given (structured) specification \(S_0\). First the library refinement is selected. The applicability of the refinement to \(S_0\) is shown by constructing a classification arrow from \(A\) to \(S_0\) which classifies \(S_0\) as having \(A\)-structure by making explicit how \(S_0\) has at least the structure of \(A\). Finally the refinement is applied by computing the pushout in the category of diagrams. The creative work lies in constructing the classification arrow \([7, 9]\).

3.3. Development of Sorting Algorithms

Our goal in this section is to show a simple example that refines a requirement spec to code by applying (1) an algorithm design refinement, (2) a datatype refinement, and (3) an expression optimization refinement. We step through the refinement of a specification for sorting a bag over an arbitrary linear order. A full specification for the import to \(\text{Sorting}\) is given in Appendix B.

```plaintext
spec Sorting is
import Bag-Seq-over-LinOrd
op sorted? : Bag, Seq \rightarrow Boolean
def sorted?(x, z) = (ordered?(z) \land x = seq-to-bag(z))
op sorting : Bag \rightarrow Seq
axiom sorted?(x, sorting(x))
end-spec
```

The following diagram serves as a roadmap of the design steps covered in the next three subsections. First a divide-and-conquer refinement is applied, then a datatype
refinement (bags implemented as sequences), then an expression simplification refinement is applied, and so on. The squares represent the application of library refinements and the main line of refinement descends directly from `Sorting`.

\[
\text{Divide-and-Conquer-0-1-2} \quad \rightarrow \quad \text{Sorting} \\
\text{Divide-and-Conquer-0-1-2-scheme} \quad \rightarrow \quad \text{Sorting-Alg}_1 \quad \leftarrow \quad \text{BAG} \\
\text{New-Expression} \quad \rightarrow \quad \text{Sorting-Alg}_2 \quad \leftarrow \quad \text{BAG-AS-SEQ} \\
\text{Equational-Simplification} \quad \rightarrow \quad \text{Sorting-Alg}_3 \\
\ldots
\]

### 3.3.1. Sorting: Algorithm Design

Before algorithm design can be applying we must have a well-defined problem to solve. `Problem-Theory` in Figure 2 expresses the abstract structure of a problem: given input datum \( x : D \), find a feasible solution \( z : R \) satisfying the problem requirement constraint: \( O(x, z) \). We call the input datum \( x \) a problem instance; note that `Problem-Theory` is intended to specify a function.

Suppose that the user first decides to apply a generic refinement for divide-and-conquer algorithms.

The principle of divide-and-conquer is to solve small problem instances by some direct means, and to solve larger problem instances by decomposing them, solving the pieces, and composing the resulting solutions. The figure to the left shows a mergesort as a generalized kind of divide-and-conquer with three cases. The bottommost square shows the familiar divide-and-conquer case: the input \( x_0 \) is decomposed into two subproblem instances, \( x_1 \) and \( x_2 \), which are solved to produce \( z_1 \) and \( z_2 \), which are in turn composed to form solution \( z_0 \). The decomposition operator here is loosely specified as the inverse of a constructor, `concat`.

The first two cases also have the decompose-solve-compose pattern where the decomposition operator is the inverse of a constructor. The datum \( \langle \rangle \) is the 0-tuple and it is treated as the output of the inverse of a constant constructor; \( \langle \rangle \) is the sole element of
the empty product sort, denoted $\text{Unit}$. The first two cases differ from the third in that the input decomposes into a subproblem that is easily solved, by the identity function on the appropriate sort.

The domain spec of a simple divide-and-conquer refinement is given in Figure 2 that abstracts from the mergesort example. It supposes three cases (indexed by 0, 1, and 2), each based on the signature of a constructor, similar to a homomorphism between algebras with three constructors. The abstraction lies in treating each operator in the example as a problem theory. Subscripted $O$'s denote problem requirement constraints; e.g. $O_{\text{Decompose}_i}$ specifies the decomposition operator in case $i$, and so on. The discriminator $p_i$ represents the condition under which $O_{\text{Decompose}_i}$ can decompose an input. The Soundness-2 axiom relates $O$, $O_{\text{Decompose}_2}$, and $O_{\text{Compose}_2}$. It asserts that if (1) nonprimitive problem instance $x_0$ can decompose into two subproblem instances $x_1$ and $x_2$, (2) subproblem instances $x_1$ and $x_2$ have feasible solutions $z_1$ and $z_2$ respectively, (3) $z_1$ and $z_2$ can compose to form $z_0$, then $z_0$ is a feasible solution to input $x_0$. Similar comments hold for the other Soundness axioms. The operator $>$ is a well-founded order on $D$ to assure termination (axioms are omitted for simplicity).

A general scheme for problem reduction theories (including divide-and-conquer) is given in [6]. The idea is to have a different first-order divide-and-conquer theory for each possible abstract constructor signature.

The codomain spec of the divide-and-conquer refinement (Figure 3) contains a schematic definition for the top-level divide-and-conquer functions and schematic requirement specifications for subalgorithms $C_0$, $C_1$, and $C_2$. Given any morphism from Divide-and-Conquer-0-1-2 and functions that satisfy the requirement specifications for the subalgorithms, the corresponding instance of the divide-and-conquer function satisfies its requirement specification [4].

There is one complication which needs explaining. Specware requires that all functions be total and deterministic; each operation in a spec has implicit axioms for existence and uniqueness of solutions. This simplifies the logic of Specware, but complicates the treatment of nondeterministic functions which arise naturally in refinement, due to the levels of abstraction. Consider, for example, an operation that decomposes a bag $b$ into an element $a$ and the remainder of the bag $b'$; i.e. an inverse of insert-bag($a, b'$). If the bags are ultimately implemented as lists, then it is natural to use first and rest to perform the decomposition. However consider two equal bags $\{1,1,2\} = \{2,1,1\}$. If $\{1,1,2\}$ is represented by $[1,1,2]$ then it will decompose into 1 and $\{1,2\}$. If $\{2,1,1\}$ is represented by $[2,1,1]$, then it will decompose into 2 and $\{1,1\}$. Hence the decomposition is not functional on bags, even though it is functional on lists.

There are various approaches to this problem. We observe however that any extra work required to make a naturally nondeterministic operator into a deterministic one is often unnecessary from the programming point of view, when the nondeterminism is of the don’t-care variety – any solution will do, so any extra work to force one unique solution is a constraint posed by the programming system rather than by the software requirements. One solution to this problem is to treat operations that may need to be nondeterministic via existential quantification. In place of decomposition operators in Figure 3 we have existentially quantified requirement constraints.

The development of a divide-and-conquer algorithm for sorting begins with the con-
spec Problem-Theory is
  sorts D, R
  op O : D, R → Boolean
end-spec

spec Divide-and-Conquer-0-1-2 is
  import DRO
  sort E
  op F : D → R
  op _ > _ : D, D → Boolean
  axioms > is a well-founded order ...
  
  op p0 : D → Boolean
  op ODecompose0 : D, Unit → Boolean
  op OCompose0 : R, Unit → Boolean
  axiom Soundness-0 is
    ODecompose0(x, ⟨⟩) ∧ OCompose0(z, ⟨⟩) → O(x, z)
  axiom discriminator-of-Decompose0 is
    p0(x) → ODecompose0(x, ⟨⟩)
  
  op p1 : D → Boolean
  op ODecompose1 : D, E → Boolean
  op OCompose1 : R, E → Boolean
  axiom Soundness-1 is
    ODecompose1(x, e) ∧ OCompose1(z, e) → O(x, z)
  axiom discriminator-of-Decompose1 is
    p1(x) → ∃(e) ODecompose1(x, e)
  
  op p2 : D → Boolean
  op ODecompose2 : D, D, D → Boolean
  op OCompose2 : R, R, R → Boolean
  axiom Soundness-2 is
    ODecompose2(x0, x1, x2) ∧ O(x1, z1) ∧ O(x2, z2) ∧ OCompose2(z0, z1, z2) → O(x0, z0)
  axiom discriminator-of-Decompose2 is
    p2(x) → ∃(x1, x2) ODecompose2(x, x1, x2) ∧ x > x1 ∧ x > x2
  
  axiom ∀(x : D) p0(x) xor p1(x) xor p2(x)
end-spec

Figure 2: A Simple Divide-and-Conquer Algorithm Theory
spec Divide-and-Conquer-0-1-2-scheme is
import Divide-and-Conquer-0-1-2
op C₀ :→ R
axiom OCompose₀(C₀, ⟨⟩)

op C₁ : E → R
axiom OCompose₁(C₁(e), e)

op C₂ : R, R → R
axiom OCompose₂(C₂(x₁, x₂), x₁, x₂)

definition of F is
axiom p₀(x) =⇒ ODecompose₀(x, ⟨⟩) ∧ F(x) = C₀
axiom p₁(x) =⇒ ∃(e)(ODEcompose₁(x, e) ∧ F(x) = C₁(e))
axiom p₂(x) =⇒ ∃(x₁, x₂)(ODEcompose₂(x, x₁, x₂)
∧ F(x) = C₂(F(x₁), F(x₂)))
end-definition

end-spec

Figure 3: Parameterized Divide-and-Conquer Algorithm

struction of a morphism from Problem-Theory to Sorting theory:

\[
\begin{align*}
D & \mapsto Bag \\
R & \mapsto Seq \\
O & \mapsto sorted?
\end{align*}
\]

Since the morphism from Problem-Theory to Divide-and-Conquer is an inclusion, we can use straightforward propagation to obtain translations for the components of Problem-Theory in Divide-and-Conquer:

\[
\begin{align*}
D & \mapsto Bag \\
R & \mapsto Seq \\
O & \mapsto sorted? \\
F & \mapsto ? \\
E & \mapsto ? \\
> & \mapsto ? \\
p₀ & \mapsto ? \\
O{Decompose₀} & \mapsto ? \\
O{Compose₀} & \mapsto ? \\
\ldots
\end{align*}
\]

To complete the classification arrow we attempt to translate the remaining operators into expressions of Sorting. Alternative translations give rise to different sorting algorithms. There are several ways to proceed. One approach is based on the choice of a set of
standard decomposition operators from a library. The tactic then uses unskolemization on the soundness axioms to derive specifications for the composition operators. This approach allows the derivation of insertion sort, mergesort, and various parallel sorting algorithms [4, 8]. A dual approach is to choose a set of standard composition operators from a library and use the soundness axioms to derive the decomposition operators (leading to selections sort, heapsort, and quicksort). We present the first approach in detail, then sketch the second.

Suppose that we choose the constructor set \{empty-bag, singleton-bag, bag-union\} as the basis for the decomposition relation on the input domain Bag. This gives us the partial signature morphism

\[
\begin{align*}
D & \mapsto Bag \\
R & \mapsto Seq \\
O & \mapsto sorted? \\
F & \mapsto sorting \\
E & \mapsto E > \\
p_0 & \mapsto empty-bag? \\
op_{Decompose0} & \mapsto \lambda(x) x = empty-bag \\
op_{Compose0} & \mapsto ? \\
p_1 & \mapsto singleton-bag? \\
op_{Decompose1} & \mapsto \lambda(x,e) x = singleton-bag(e) \\
op_{Compose1} & \mapsto ? \\
p_2 & \mapsto nonsingleton-bag? \\
op_{Decompose2} & \mapsto \lambda(x_0,x_1,x_2) x_0 = bag-union(x_1,x_2) \\
op_{Compose2} & \mapsto ? \\
\end{align*}
\]

The soundness axiom

\[
\forall(x_0,x_1,x_2 : D)\forall(z_0,z_1,z_2 : R)
(O_{Decompose2}(x_0,x_1,x_2) \land O(x_1,z_1) \land O(x_2,z_2) \land O_{Compose2}(z_0,z_1,z_2)
\implies O(x_0,z_0))
\]

cannot be translated into Sorting because \(O_{Compose2}\) has no translation yet. However, a technique called unskolemization allows us to use inference tools to deduce a suitable translation for \(O_{Compose2}\) [7]. Unskolemizing operator symbol \(O_{Compose2}\) replaces the occurrence of \(O_{Compose2}\) by a fresh existentially quantified variable in the scope of the quantifiers for \(z_0, z_1, z_2\):

\[
\forall(z_0,z_1,z_2 : R) \exists(y : Boolean) \forall(x_0,x_1,x_2 : D)
(O_{Decompose2}(x_0,x_1,x_2) \land O(x_1,z_1) \land O(x_2,z_2) \land y \implies O(x_0,z_0)).
\]

This formula has the same satisfiability properties as the original and it can be translated to Sorting via the partial morphism yielding:

\[
\forall(z_0,z_1,z_2 : Seq) \exists(y : Boolean) \forall(x_0,x_1,x_2 : Bag)
(x_0 = bag-union(x_1,x_2) \land sorted?(x_1,z_1) \land sorted?(x_2,z_2) \land y \implies sorted?(x_0,z_0)).
\]
A straightforward proof of this formula in *Sorting* finds a witness for \( y \) which results in a translation for \( O_{\text{Compose2}} \):

\[
\text{sorted?}(x_0, z_0) = \text{by def of sorted?} \\
\quad \text{ordered}(z_0) \land x_0 = \text{seq-to-bag}(z_0) \\
\quad = \text{by assumption } x_0 = \text{bag-union}(x_1, x_2) \\
\quad \text{ordered}(z_0) \land \text{bag-union}(x_1, x_2) = \text{seq-to-bag}(z_0) \\
\quad = \text{by assumption } x_i = \text{seq-to-bag}(z_i), \ i = 1, 2 \\
\quad \text{ordered}(z_0) \land \text{bag-union(} \text{seq-to-bag}(z_1), \text{seq-to-bag}(z_2)\text{)} = \text{seq-to-bag}(z_0).
\]

which can then unify with the assumption \( y \), which must be a term over variables \( z_0, z_1, z_2 \). More generally, we pick up as part of the witness any assumptions that are expressed over the variables \( z_0, z_1, z_2 \), in this case yielding

\[
O_{\text{Compose2}} \mapsto \{ \lambda(z_0, z_1, z_2) \text{ ordered}(z_1) \land \text{ordered}(z_2) \\
\quad \implies \text{ordered}(z_0) \\
\quad \land \text{bag-union(} \text{seq-to-bag}(z_1), \text{seq-to-bag}(z_2)\text{)} = \text{seq-to-bag}(z_0).\}
\]

This is, of course, a specification for a merge operation. If we take this as the translation of \( O_{\text{Compose2}} \), then we know that the Soundness-2 axiom translates to a theorem in *Sorting* by construction.

The remaining steps in constructing this classification arrow are similar. We unskolemize \( O_{\text{Compose1}} \) in Soundness-1 and then derive a witness:

\[
\text{sorted?}(x, z) = \text{by def of sorted?} \\
\quad \text{ordered}(z) \land x = \text{seq-to-bag}(z) \\
\quad = \text{using assumption } x = \text{singleton-bag}(e) \\
\quad \text{ordered}(z) \land \text{singleton-bag}(e) = \text{seq-to-bag}(z) \\
\quad = \text{Bag-and-Seq-Conv axiom} \\
\quad \text{ordered}(z) \land \text{seq-to-bag(} \text{singleton-seq}(e)\text{)} = \text{seq-to-bag}(z) \\
\quad \iff \text{Leibnitz: equality is a congruence} \\
\quad \text{ordered}(z) \land \text{singleton-seq}(e) = z \\
\quad = \text{simplifying using ordered(} \text{singleton-seq}(e)\text{)} = \text{true} \\
\quad \text{singleton-seq}(e) = z.
\]

yielding the translation

\[
O_{\text{Compose1}} \mapsto \lambda(z, e) \text{ singleton-seq}(e) = z.
\]

We can unskolemize \( O_{\text{Compose0}} \) in Soundness-0 and then derive a witness:

\[
\text{sorted?}(x, z) = \text{def of sorted?} \\
\quad \text{ordered}(z) \land x = \text{seq-to-bag}(z) \\
\quad = \text{assumption } x = \text{empty-bag}
\]
\[
\text{ordered}(z) \land \text{empty-bag} = \text{seq-to-bag}(z) = \text{Bag-and-Seq-Conv axiom}
\]
\[
\text{ordered}(z) \land \text{seq-to-bag}(\text{empty-seq}) = \text{seq-to-bag}(z) = \text{Leibnitz: equality is a congruence}
\]
\[
\text{ordered}(z) \land \text{empty-seq} = z = \text{simplifying using Sorting axiom ordered}(\text{empty-seq}) = \text{true}
\]

Empty-seq = z.

yielding the translation

\[
O_{\text{Compose}0} \mapsto \lambda (z) \text{ empty-seq} = z.
\]

The classification arrow is now complete:

\[
\begin{align*}
D & \mapsto \text{Bag} \\
R & \mapsto \text{Seq} \\
O & \mapsto \text{sorted?} \\
F & \mapsto \text{sorting} \\
E & \mapsto E \\
> & \mapsto \text{bag-wfgt} \\
p_0 & \mapsto \text{empty-bag?} \\
O_{\text{Decompose}0} & \mapsto \lambda (x) \; x = \text{empty-bag} \\
O_{\text{Compose}0} & \mapsto \lambda (z) \; z = \text{empty-seq} \\
p_1 & \mapsto \text{singleton-bag?} \\
O_{\text{Decompose}1} & \mapsto \lambda (x, e) \; x = \text{singleton-bag}(e) \\
O_{\text{Compose}1} & \mapsto \lambda (z, e) \; z = \text{singleton-seq}(e) \\
p_2 & \mapsto \text{nonsingleton-bag?} \\
O_{\text{Decompose}2} & \mapsto \lambda (x_0, x_1, x_2) \; x_0 = \text{bag-union}(x_1, x_2) \\
O_{\text{Compose}2} & \mapsto \lambda (z_0, z_1, z_2) \; \text{ordered}(z_1) \land \text{ordered}(z_2) \implies \text{ordered}(z_0) \\
& \land \text{seq-to-bag}(z_0) = \text{bag-union}(\text{seq-to-bag}(z_1), \text{seq-to-bag}(z_2))
\end{align*}
\]

Note that we have a classification arrow that translates a symbol to an expression (rather than a symbol). This commonly occurring situation is treated by extending the codomain diagram with definitional extensions as necessary. In the sequel we will assume this treatment whenever the need for a symbol-to-expression translation arises.

We can compute the pushout

\[
\begin{align*}
\text{Divide-and-Conquer-0-1-2} & \mapsto \text{Sorting} \\
\text{Divide-and-Conquer-0-1-2-scheme} & \mapsto \text{Sorting-Alg}_1
\end{align*}
\]

to obtain a refinement of Sorting that contains a definition for a mergesort algorithm, shown in Figure 4.

Example – sketch of a derivation for Quicksort:
spec Sorting-Alg₁ is
  import Bag-Seq-over-LinOrd

  op sorted? : Bag, Seq → Boolean
  definition sorted?(x, z) = (ordered?(z) ∧ x = seq-to-bag(z))

  op sorting : Bag → Seq
  theorem sorted?(x, sorting(x))
  definition of sorting is
    axiom empty-bag?(x) ⟹ (x = empty-bag ∧ sorting(x) = C₀)
    axiom singleton-bag?(x) ⟹ ∃(e)(x = singleton-bag(e) ∧ sorting(x) = C₁(e))
    axiom nonsingleton-bag?(x)
      ⟹ ∃(x₁, x₂)(x = bag-union(x₁, x₂)
      ∧ sorting(x) = C₂(sorting(x₁), sorting(x₂)))
  end-definition

  op C₀ : → Seq
  axiom C₀ = empty-seq

  op C₁ : E → Seq
  axiom C₁(e) = singleton(e)

  op C₂ : Seq, Seq → Seq
  axiom ordered?(z₁) ∧ ordered?(z₂)
    ⟹ (ordered?(C₂(z₁, z₂))
    ∧ seq-to-bag(C₂(z₁, z₂)) = bag-union(seq-to-bag(z₁), seq-to-bag(z₂)))
  end-spec

Figure 4: Divide-and-Conquer Sorting Algorithm
The derivation of a variant of quicksort is dual to the derivation for mergesort: rather than select a simple set of constructors as the basis for the decomposition operations, we select a simple set of constructors as a basis for the composition operators. In particular if we return to the stage of filling in the partial morphism

\[
\begin{align*}
D & \mapsto Bag \\
R & \mapsto Seq \\
O & \mapsto sorted? \\
F & \mapsto sorting \\
E & \mapsto E \\
> & \mapsto bag-wfgt \\
p_0 & \mapsto ? \\
O_{Decompose0} & \mapsto ? \\
O_{Compose0} & \mapsto ? \\
\end{align*}
\]

and we choose the set of sequence constructors \{empty-seq, singleton-seq, concat\} as a basis for composition, we get

\[
\begin{align*}
D & \mapsto Bag \\
R & \mapsto Seq \\
O & \mapsto sorted? \\
F & \mapsto sorting \\
E & \mapsto E \\
> & \mapsto bag-wfgt \\
p_0 & \mapsto ? \\
O_{Decompose0} & \mapsto ? \\
O_{Compose0} & \mapsto \lambda(z) z = empty-seq \\
p_1 & \mapsto ? \\
O_{Decompose1} & \mapsto ? \\
O_{Compose1} & \mapsto \lambda(z,e) z = singleton-seq(e) \\
p_2 & \mapsto ? \\
O_{Decompose2} & \mapsto ? \\
O_{Compose2} & \mapsto \lambda(z_0, z_1, z_2) z_0 = concat(z_1, z_2) \\
\end{align*}
\]

Now we use the Soundness axioms to deduce translations for the decomposition requirement constraints. The most interesting case is unskolemizing operator symbol \(O_{Decompose2}\) in axiom Soundness-2:

\[
\forall(x_0, x_1, x_2 : D) \exists(y : Boolean) \forall(z_0, z_1, z_2 : R)
(y \land O(x_1, z_1) \land O(x_2, z_2) \land O_{Compose2}(z_0, z_1, z_2)
\implies O(x_0, z_0)).
\]

This formula can be translated via the partial morphism yielding:

\[
\forall(x_0, x_1, x_2 : Bag) \exists(y : Boolean) \forall(z_0, z_1, z_2 : Seq)
(y \land sorted?((x_1, z_1)) \land sorted?((x_2, z_2)) \land z_0 = concat(z_1, z_2)
\implies sorted?((x_0, z_0)))
\]

20
A straightforward proof of this formula in \textit{Sorting}:
\[
\text{sorted?}(x_0, z_0) = \begin{align*}
&= \text{by def of sorted?} \\
&= \text{ordered}(z_0) \land x_0 = \text{seq-to-bag}(z_0) \\
&= \text{by assumption } z_0 = \text{concat}(z_1, z_2) \\
&= \text{ordered(concat}(z_1, z_2)) \land x_0 = \text{seq-to-bag(concat}(z_1, z_2)) \\
&= \text{distributing using axioms from Sorting and Bag-and-Seq-Conv} \\
&= \text{ordered}(z_1) \land \text{all-le}(z_1, z_2) \land \text{ordered}(z_2) \\
&= \text{simplifying using seq-to-bag}(z_i) = x_i, i = 1, 2 \\
&= \text{all-le}(x_1, x_2) \land x_0 = \text{bag-union}(x_1, x_2)
\end{align*}
\]

results in a translation for \(O_{\text{Decompose}2}\):
\[
\text{O}_{\text{Decompose}2} \mapsto \lambda(x_0, x_1, x_2) \text{ all-le}(x_1, x_2) \land x_0 = \text{bag-union}(x_1, x_2)
\]

That is, each element of bag \(x_1\) is less-than-or-equal to each element of bag \(x_2\) and \(x_0\) decomposes into two exhaustive subbags of itself. This is, of course, a specification for a partition operation in a quicksort. If we take this as the translation of \(O_{\text{Decompose}2}\), then we know that the soundness axiom translates to a theorem in \textit{Sorting-theory} by construction. The other two cases are similar to the base cases in mergesort.

The divide-and-conquer theory allows a pleasing derivational symmetry between algorithms with simple decomposition operations and complex composition operations and dually, algorithms with simple composition operations and complex decomposition operations.

### 3.3.2. Sorting: Datatype Refinement

Next, suppose that the user decides to bring the mergesort algorithm closer to implementation by refining bags into sequences. This datatype refinement was discussed in Section 2.4.2. and it is presented in detail in Appendix A. The classification arrow from BAG to \textit{Sorting} is essentially an inclusion, comprised of the identity morphisms from BAG specs to corresponding specs in the diagram of the \textit{Sorting} spec.

The classification arrow together with the pushout is shown in Figure 5. The two cocone arrows in the figure are shown with dashes to help distinguish them. Also, to save space the figure abbreviates \textit{Bag-as-Seq} to \textit{BaS}.

The pushout gives us a refinement of \textit{Sorting}, shown in Figure 6. The effect on the mergesort definition arises mainly from the translations of the \(BtoS\) \textit{Bag} morphism in Section 2.4.2. Note that not all bag operators are directly in the image of the classification arrow. In particular, the operators \textit{all-le} and \textit{seq-to-bag} are extensions to bag theory added during the construction of sorting theory. The fact that they are defined however
means that the pushout will translate the definitions too, so they remain defined after translation. If some of the auxiliary sorts or operators on the datatype are not in the image of the classification arrow, then the pushout will simply translate whatever constraints there are on such sorts and operators. If the constraints are definitions, then the translation will be definitional, otherwise the symbols will be constrained but not necessarily uniquely, and further refinement will be required later.

This datatype refinement mainly serves to bring the spec closer to the programming language level. Other datatype refinements have a more dramatic effect by refining to a complicated data structure that efficiently implements the more abstract type. The same machinery applies. Leverage on the programming task will be most apparent when a relatively small domain specification refines to a relatively large codomain spec – the extra bits of information in the codomain are added to the user’s spec essentially for free once the classification arrow has been constructed. Some examples would include Boolean expressions refining to BDDs, and sets over a fixed finite universe refining to bit vectors or hash tables, and finite relations refining to B-trees.
spec Sorting-Alg2 is
import Bag-as-Seq-Seq-over-LinOrd
op sorted? : Bag-as-Seq, Seq → Boolean
def sorted?(x, z) = (ordered?(z) ∧ x = seq-to-bag(z))

op sorting : Bag-as-Seq → Seq
definition of sorting is
   axiom bag-empty?(x) → (x = bag-empty ∧ sorting(x) = C0)
   axiom bag-singleton?(x) → ∃(e)(x = bag-singleton(e) ∧ sorting(x) = C1(e))
   axiom bag-nonsingleton?(x)
      → ∃(x1, x2)(x = bag-union(x1, x2)
                       ∧ sorting(x0) = C2(sorting(x1), sorting(x2)))
end-definition

op C0 :→ Seq
axiom C0 = empty-seq

op C1 : E → Seq
axiom C1(e) = singleton(e)

op C2 : Seq, Seq → Seq
axiom ordered?(z1) ∧ ordered?(z2)
   → (ordered?(C2(z1, z2))
      ∧ seq-to-bag(C2(z1, z2)) = bag-union(seq-to-bag(z1), seq-to-bag(z2)))
end-spec

Figure 6: Sorting Algorithm after Datatype Refinement

3.3.3. Sorting: Expression Optimization

After performing algorithm design and datatype refinement, there are often opportunities for optimizing various subexpressions. Transformation rules and optimization tactics, such as equational rewriting, context-dependent simplification [5], finite differencing [3], partial evaluation, and other more specialized techniques can result in dramatic improvements in the complexity of the final implementation. We show how at least some optimization techniques can be captured as refinements, and can be applied in the same manner as refinements that express algorithm design and datatype refinement knowledge.

The general approach to representing expression optimization techniques is a two stage process: first, an equation is deduced with the expression-to-be-simplified on the left-hand-side, and second, the spec S is reformulated to exploit the new theorem. The applicable metatheorem here is: if spec S1 has theorem e1 = e2 and S2 is the same spec as S1 except that some occurrences of e1 are replaced by e2, then S1 and S2 are isomorphic specifications. The metatheorem justifies a substitution operator on specs
that creates a spec and the isomorphism between them.

Equational rewriting is a basic optimization technique that uses equational inference in the user’s spec to deduce an equal term. However, to determine which side is simpler requires a metric. The approach used in KIDS [5] and in Specware/Designware is to assign a weight to each operator, measuring the complexity of an expression by summing the weights of each operator occurrence.

Here is a refinement to express equational simplification:

```plaintext
spec Expression is
  sorts D, Q
  op expr : D → Q
end-spec

spec New-Expression is
  import Expression
  op new-expr : D → Q
  axiom expr(x) = new-expr(x)
end-spec

spec Equational-Simplification is
  import New-Expression
  theorem expr(x) = new-expr(x)
end-spec
```

Equational-Simplification (ES) is a degenerate case of our approach to representing expression optimization techniques. The presence of the theorem in ES is required in order that it be included in the pushout spec when the refinement is applied.

Context-dependent simplification is a generalization of equational simplification. Suppose that we wish to simplify expression expr in some context C within spec S. To simplify expr, we want to use not only the equations and other theorems of the spec S, but also whatever contextual properties hold when expr is evaluated. For example, when simplifying the else-branch of a conditional, we can assume the negation of the test of the conditional.

```plaintext
spec Expression-and-Context is
  import Expression
  op C : D → Boolean
  op new-expr : D → Q
  axiom C(x) → expr(x) = new-expr(x)
end-spec

spec Context-Dependent-Simplification is
  import Expression-and-Context
  theorem C(x) → expr(x) = new-expr(x)
end-spec
```
Finite Differencing [3] is a less basic optimization and it has nondegenerate structure in the codomain of its representation as a refinement. The idea in finite differencing is to replace an expensive expression in a loop by incremental computation, making use of some extra variables. The following refinement expresses the correspondence between these concepts and the symbols in the refinement: \( e \) is the expensive expression, which may occur in the definition of operator \( f \); \( h \) is the body of the definition of \( f \). \( g \) is a new operator which is an abstraction of \( f \) with an extra parameter \( c \) that is maintained to satisfy the invariant \( c = e(x) \). The theorem asserts equality between \( f \) and \( g \) when \( g \) is invoked with an argument that satisfies the invariant. After developing the classification arrow and pushing out, the substitution (meta)operator replaces occurrences of \( f \) with \( g \). To realize the implicit incremental benefits of this change, context-dependent simplification must be applied to exploit the local invariant \( c = e(x) \).

We show how the equational simplification refinement works by simplifying the clauses of the definition of sorting in \( \text{Sorting-Alg}_2 \). The simplification often serves a larger goal. Following algorithm design there are usually opportunities to simplify expressions due to the juxtaposition of subexpressions from different sources that have been assembled by means of a definitional scheme. More pertinent here, when datatype refinement is performed and the codomain has a constructed sort, then simplification can reexpress definitions in terms of the base sorts of the construction. The definitional axioms of sorting in \( \text{Sorting-Alg}_2 \) are expressed in terms of a quotient sort construction

\[
\text{sort-axiom Bag-as-Seq} = \text{Seq/perm}?
\]

Simplification will have the effect of reexpressing the axioms in terms of sequences, using coercion functions.

We show the simplification of the singleton case since it most clearly exemplifies the essential reasoning pattern. The classification arrow can be partially filled in once a user has selected an expression to simplify:
To obtain a translation for new-expr, we unskolemize it in the New-Expression axiom, then translate the result and find a witness in Sorting theory as shown in Figure 7. The proof draws on axioms from Bag-as-Seq and Seq and the quantifier change theorems:

\[
(\forall (x : D/ \equiv) \ A(x)) = (\forall (y : D) \ A(q(y))) \quad \text{where } q = \text{quotient}(\equiv)
\]

\[
(\forall (x : D/p?) \ A(r(x))) = (\forall (y : D)(p?(y) \implies A(y))) \quad \text{where } r = \text{relax}(p?)
\]

The completed the classification arrow is:

\[
\begin{align*}
D & \mapsto Unit \\
Q & \mapsto Boolean \\
expr & \mapsto \forall (x : \text{Bag-as-Seq})(\text{bag-singleton}? (x) \implies \\
& \quad \exists (e) (x = \text{bag-singleton}(e) \land \text{sorting}(x) = C_1(e))) \\
new-expr & \mapsto ?
\end{align*}
\]

The effect of computing the pushout of the refinement and the classification arrow is to add the theorem asserting equality of these two expressions in context. We then apply a substitution (meta)operator that creates an isomorphic spec. After simplifying all three axioms of sorting, these refinements have the effect shown in Figure 8. The special form restrict coerces a value to a subsort, provided that the subsort predicate holds in context. Sorting-Alg4, also shown in Figure 8 shows the effect of aggregating the clauses of the sorting definition into a conditional using another refinement.

### 3.3.4. Sorting: Summary

One remaining step is to design a definition for C2. This spec can be refined either by applying divide-and-conquer to create one of several possible simple merge operators: the straightforward one is the usual linear time sequential merge [4]; an alternate derivation yields the log time parallel merge underlying Batcher’s Sort [8]. Another way to synthesize a definition for C2 is to reduce it to “legacy code”; that is, an existing library routine that can satisfy the C2 requirement specification. This reduction process uses a general mechanism called connections between theories [7].

The code generation process uses library refinements from the specification language to a programming language (currently in Specware these are CommonLISP and C++).
∀(x : Bag-as-Seq)(bag-singleton?(x)
    → ∃(e)(x = bag-singleton(e) ∧ sorting(x) = C₁(e)))

= quantifier change via q : Seq → Bag-as-Seq = quotient(perm?)

∀(y : Seq)(bag-singleton?(q(y))
    → ∃(e)(q(y) = bag-singleton(e) ∧ sorting(q(y)) = C₁(e)))

= simplifying, using the defs of bag-singleton? and bag-singleton

∀(y : Seq)(singleton-seq?(y)
    → ∃(e)(q(y) = q(singleton-seq(e)) ∧ sorting(q(y)) = C₁(e)))

= quantifier change via the subsort coercion
r₁ : 1-Seq → Seq = relax(singleton-seq?)

∀(z : 1-Seq)∃(e)(q(r₁(z)) = q(singleton-seq(e)) ∧ sorting(q(r₁(z))) = C₁(e))

= unifying with q(r₁(w)) = q(singleton-seq(singleton-seq-inv(w)))
  which is inferred from the inverse axiom
  r₁(w) = singleton-seq(singleton-seq-inv(w))

∀(z : 1-Seq) sorting(q(r₁(z))) = C₁(singleton-seq-inv(z))

= unfolding the def of C₁

∀(z : 1-Seq) sorting(q(r₁(z))) = singleton-seq(singleton-seq-inv(z))

= simplifying, again using the inverse axiom

∀(z : 1-Seq) sorting(q(r₁(z))) = r₁(z)

= quantifier change in reverse, using subsort coercion r₁ : 1-Seq → Seq

∀(y : Seq) singleton-seq?(y) → sorting(q(y)) = y

Figure 7: Simplification of a sorting axiom
spec Sorting-Alg₃ is
  import Bag-as-Seq-Seq-over-LinOrd
...
  op sorting : Bag-as-Seq → Seq
  theorem sorted?(x, sorting(x))
  definition of sorting is
    axiom empty-seq?(x) ⇒ sorting(q(x)) = x
    axiom singleton-seq?(x) ⇒ sorting(q(x)) = x
    axiom nonsingleton-seq?(x) ⇒
      let (x₁ = q(left-split(restrict(x))),
           x₂ = q(right-split(restrict(x))))
      sorting(q(x)) = C₂(sorting(x₁), sorting(x₂))
  end-definition
...
end-spec

spec Sorting-Alg₄ is
  import Bag-as-Seq-Seq-over-LinOrd
...
  op sorting : Bag-as-Seq → Seq
  theorem sorted?(x, sorting(x))
  def sorting(q(x)) =
    (if empty-seq?(x) then x
     elseif singleton-seq?(x) then x
     else (let (x₁ = q(left-split(restrict(x))),
               x₂ = q(right-split(restrict(x))))
          C₂(sorting(x₁), sorting(x₂)))
  end-definition
...
end-spec

Figure 8: Sorting Algorithm after Simplifications
4. Domain-Specific Software Development

The sorting example shows how to exploit domain-independent design knowledge that is uniformly represented by refinements. In this section the Planware system provides examples of using refinements to represent domain-specific design knowledge.

Planware [1] is a synthesis system that is specialized to the production of scheduling algorithms. It extends Specware/Designware with libraries of theories and refinements about scheduling and with a specialized tactic for controlling the application of such design knowledge. Planware applies library refinements to transform information from the user into a formal requirement specification, which is typically thousands of line of specification text. The refinement to code via algorithm design, datatype refinement, and expression optimization is completely automatic.

4.1. Constructing a Requirement Specification

Planware provides an answer to the question of how to help automate the acquisition of requirements from the user and to assemble a formal requirement specification for the user. The key idea is to focus on a narrow well-defined class of programs and to prebuild an abstract specification that covers the class. Interaction with the user is only required in order to obtain the refinement from the abstract spec to a specification of the requirements of the user’s particular problem.

The scheduling problem in general is to compute a set of reservations on a given set of resources in order to accomplish a given set of tasks subject to certain constraints. Optionally we may want to optimize an objective function. A sketch of an abstract scheduling specification follows, where the import Scheduling-Import combines specifications for Time, Quantity, Task, Resource, Reservation, Set-of-Task, Set-of-Resource, Set-of-Reservation:

```plaintext
spec Task
  sort Task
end-spec

spec Resource
  sort Resource
end-spec

spec Abstract-Scheduling is
  imports Scheduling-Import
  sorts Reservation = Resource, Task, Time
      Schedule = Set-of-Reservation
  op scheduler : Set-of-Task, Set-of-Resource -> Schedule

  op all-tasks-scheduled? : Set-of-Task, Schedule -> Boolean
  def all-tasks-scheduled?(tasks, schedule)
    = ∀(tsk)(tsk in tasks = ∃!(rsvn)(rsvn in schedule ∧ task(rsvn) = tsk))
  axiom all-tasks-scheduled?(tasks, scheduler(tasks, resources))
```
This specifies an abstract scheduler whose input is a set of tasks (each with unspecified structure) and a set of resources (each with unspecified structure). Two generic problem constraints are imposed on the output of the Scheduler operator: all-tasks-scheduled? asserts that it must schedule all (and only) the input tasks, and only-given-resources-used? asserts that it uses only the supplied resources. So, for example, if the input is 100 cargo items to fly on 10 aircraft, then the schedule generated by Scheduler must schedule all 100 items (and no more) and use only the 10 given aircraft.

We briefly describe two domain-specific refinements that are applied to refine this abstract scheduling spec to a spec for a particular problem, such as a transportation problem, a power plant maintenance problem, or a processor scheduling problem.

Figure 9 depicts a taxonomy of resource theories that covers many commonly occurring types of resources. Abstract Resource theory refines into Consumable resources (e.g. fuel, crew duty time) and Reusable resources (e.g. aircraft, trucks, parking lots, printers). Reusable resources further refine into Synchronously Reusable resources (e.g. ships, aircraft) where all reservations of a resource instance must be synchronized (e.g. passenger reservations on a particular airline flight are all for the same start and finish time). Synchronously Reusable resources further refine into Transportation resources (which have spatial translation between starting and ending points), and Unit Capacity resources, which can handle at most one reservation at a time.

The axiomatic formulation has several benefits over an object-oriented classification hierarchy. Both provide the signature of the relevant operations on the resource. A specification for a resource type also provides the axiomatic characterization of all properties
spec Semilattice-Structured-Attribute is
sorts Task, Attr-sort
  op attribute : Task → Attr-sort
  op rsvn-attr : Reservation → Attr-sort
  op le : Attr-sort, Attr-sort → Boolean
axioms ... le is a bounded semilattice ...
end-spec

spec Create-Constraint-from-Task-Attribute is
import Semilattice-structured-attribute
  op consistent-attr : Schedule → Boolean
  def consistent-attr(sched) =
    \forall (rsvn, tsk) (rsvn ∈ sched ∧ tsk in tasks(rsvn))
      \rightarrow attr(tsk) le rsvn-attr(rsvn))
  axiom consistent-attr(scheduler(tasks, resources))
end-spec

Figure 10: Refinement for Constructing Requirement Constraints on a Scheduler

shared by instances of the resource class. Furthermore, important theorems that provide
the basis for important algorithmic concepts can be recorded in the resource spec.

To particularize the abstract scheduling specification, Planware requests the user to
select a resource theory from the taxonomy. The composed arrow from Resource theory
to the selected theory is the arrow used for refinement.

In the next step, the user interacts with a spreadsheet-like interface that simultane-
ously gathers information about task attributes and how they are ordered as a semilattice
– this information is critical for algorithm design later. We skip the details of this pro-
cess in order to focus on how task attributes are lifted into requirement constraints on
Scheduler (see [1]). As an example, consider the release-time attribute on Task which
we suppose the user has selected as a lower bound on the start-time attribute which is
required on reservations of a transportation resource. A domain-specific refinement is
used to create problem requirement constraints is shown in Figure 10:

The codomain of the refinement creates a new predicate which is asserted as a re-
quirement constraint on Scheduler. The axiom asserts that the scheduler must produce
an output that satisfies this new constraint. From context, Planware automatically con-
structs the classification arrow:

\[
\begin{align*}
  Task & \rightarrow Task \\
  Attr-sort & \rightarrow Time \\
  attribute & \rightarrow release-time \\
  rsvn-attr & \rightarrow start-time \\
  le & \rightarrow \le
\end{align*}
\]

The pushout then automatically adds the new structure, including the requirement con-
straint on Scheduler, to the Scheduling specification:
spec Abstract-Scheduling1 is
  imports Scheduling-Import
  sorts Reservation = Resource, Task, Time
  Schedule = Set-of-Reservation
  op scheduler : Set-of-Task, set-of-Resource → Schedule

  op all-tasks-scheduled? : Set-of-Task, Schedule → Boolean
  def all-tasks-scheduled?(tasks, schedule)
     = ∀(tsk)(tsk in tasks = ∃!(rsvn)(rsvn in schedule ∧ task(rsvn) = tsk))
  axiom all-tasks-scheduled?(tasks, scheduler(tasks, resources))

  op only-given-resources-used? : Set-of-Resource, Schedule → Boolean
  def only-given-resources-used?(resources, schedule)
     = ∀(rsvn)(rsvn in schedule ⇒ resource(rsvn) in resources)
  axiom only-given-resources-used?(resources, Scheduler(tasks, resources))

  op rsvn-start-time : Reservation → Time
  op consistent-start-time : Schedule → Boolean
  def consistent-start-time(sched) =
     ∀(rsvn, tsk)(rsvn in sched ∧ tsk in tasks(rsvn)
     ⇒ release-time(tsk) ≤ rsvn-start-time(rsvn))
  axiom consistent-start-time(scheduler(tasks, resources))

end-spec

This new requirement constraint on the scheduler asserts that the actual start time of a reservation must be no earlier than the release date of any task in the reservation. Applying this domain-specific yet abstract refinement to all attributes of Task gathered from dialog with the user allows Planware to automatically generate the requirements constraints on Scheduler (without the user having to know, see, or write advanced mathematics).

4.2. Datatype Refinement and Problem Reformulation

Datatype refinement plays several important roles in Planware. We briefly mention one use at the problem formulation stage that is critical to the development of good algorithms at a later stage.

\[
\begin{array}{c}
Set(\text{Resource} \times \text{Task} \times \text{Time}) \\
\downarrow \\
Map(\text{Resource}, Set(\text{Resource} \times \text{Task} \times \text{Time})) \\
\downarrow \\
Map(\text{Resource}, \text{Ordered-Seg}(\text{Resource} \times \text{Task} \times \text{Time}))
\end{array}
\]

At the most abstract level in Planware, a schedule is presented as a set of reservations
(which in turn are effectively tuples). This abstraction is well-suited for gathering and formalizing user requirements, but quite poor if implemented in a straightforward or naive way (e.g. as lists). Planware exploits refinements shown above, where the second refinement exploits the linear ordering of time to refine a set to a sorted sequence (ordered by increasing time). The effect is to refine schedule-as-a-relation into schedule-as-a-Gantt-chart (or time-function) – a common representation in scheduling algorithms. The main benefit is that several of the output constraints on the scheduler function now simplify tremendously. Put another way, the refined data structures satisfy some of the constraints implicitly, they are built-in, so they need not be explicit.

A clear example of this phenomenon arises in the \( k \)-queens problem. A straightforward specification of this problem would define a \( k \times k \) bit matrix to represent placement of queens on a \( k \times k \) chessboard. In this representation we have four constraints to satisfy: no-two-queens-per-row, no-two-queens-per-column, no-two-queens-per-ascending-diagonal, no-two-queens-per-descending-diagonal. The constraint that there is at most one queen per column allows us to refine the bit matrix to a sequence where the \( i^{th} \) entry represents the row number of the queen in that column, if there is one. The effect is to cut down the space of possible solutions by building in the constraint no-two-queens-per-column. Furthermore, the specification of the queens problem can be condensed since the no-two-queens-per-column constraint simplifies away in the refined representation.

This phenomenon points out another benefit of a refinement approach: it supports users in exploring alternative formulations of their problem. A complicated (i.e. lots of information) but efficient (i.e. tighter representation of the problem so that there’s less junk in the search space) representation can be derived from a simpler, more understandable formulation. Oftentimes in current software development approaches the inertia of the lots-of-bits formulation may discourage users from backing up and rethinking/reformulating their approach. The refinement approach encourages a carefully staged design process wherein the user first expresses the most abstract essential requirements on the desired software, and only then begins to explore decisions about formulation, algorithms, architectures, data structures, and so on.

5. Scaling up

The process of refining specification \( S_0 \) described above has three basic steps:

1. select a refinement \( A \Rightarrow B \) from a library,
2. construct a classification arrow \( A \Rightarrow S_0 \), and
3. compute the pushout \( S_1 \) of \( B \leftarrow A \Rightarrow S_0 \).

The resulting refinement is the cocone arrow \( S_0 \Rightarrow S_1 \). This basic refinement process is repeated until the relevant sorts and operators of the spec have sufficiently explicit definitions that they can be easily translated to a programming language, and then compiled.

In this section we address the issue of how this basic process can be further developed in order to scale up as the size and complexity of the library of specs and refinements
The first key idea is to organize libraries of specs and refinements into taxonomies. The second key idea is to support tactics at two levels: theory-specific tactics for constructing classification arrows, and task-specific tactics that compose common sequences of the basic refinement process into a larger refinement step.

### 5.1. Design by Classification: Taxonomies of Refinements

A productive software development environment will have a large library of reusable refinements, letting the user (or a tactic) select refinements and decide where to apply them. The need arises for a way to organize such a library, to support access, and to support efficient construction of classification arrows. A library of refinements can be organized into taxonomies where refinements are indexed on the nodes of the taxonomies, and the nodes include the domains of various refinements in the library. The taxonomic links are refinements, indicating how one refinement applies in a stronger setting than another.

Figure 11 sketches a taxonomy of abstract datatypes for collections. Details are given in Appendix A. The arrows between nodes express the refinement relationship; e.g. the morphism from Proto-Seq to Proto-Bag is an extension with the axiom of commutativity applied to the join constructor of Proto-Seqs. Datatype refinements are indexed by the specifications in the taxonomy; e.g. a refinement from (finite) bags to (finite) sequences is indexed at the node specifying (finite) bag theory. Figure 12 shows the beginnings of
a taxonomy of expression optimization refinements that includes the ones presented in Section 3.3.3. Figure 13 shows a taxonomy of algorithm design theories. The refinements indexed at each node correspond to (families of) program schemes. The algorithm theory associated with a scheme is sufficient to prove the consistency of any instance of the scheme.

Nodes that are deeper in a taxonomy correspond to specifications that have more structure than those at shallower levels. Generally, we wish to select refinements that are indexed as deeply in the taxonomy as possible, since the maximal amount of structure in the requirement specification will be exploited. In the algorithm taxonomy, the deeper the node, the more structure that can be exploited in the problem, and the more problem-solving power that can be brought to bear. Roughly speaking, narrowly scoped but faster algorithms are deeper in the taxonomy, whereas widely applicable general algorithms are at shallower nodes.

Two problems arise in using a library of refinements: (1) selecting an appropriate refinement, and (2) constructing a classification arrow. If we organize a library of refinements into a taxonomy, then the following ladder construction process provides incremental access to applicable refinements, and simultaneously, incremental construction of classification arrows.

The process of incrementally constructing a refinement is illustrated in the ladder construction diagram to the left. The left side of the ladder is a path in a taxonomy starting at the root. The ladder is constructed a rung at a time from the top down. The initial interpretation from $A_0$ to $Spec_0$ is often simple to construct. The rungs of the ladder are constructed by a constraint solving process that involves user choices, the propagation of consistency constraints, calculation of colimits, and constructive theorem proving [7, 9]. Generally, the rung construction is stronger than a colimit – even though a co-cone is being constructed. The intent in constructing $I_i : A_i \rightarrow Spec_i$ is that $Spec_i$ has sufficient defined symbols to serve as the codomain. In other words, the implicitly defined symbols in $A_i$ are translated to explicitly defined symbols in $Spec_i$.

Once we have constructed a classification arrow $A_n \Rightarrow Spec_0$ and selected a refinement $A_n \Rightarrow B_n$ that is indexed at node $A_n$ in the taxonomy, then constructing a refinement of $Spec_0$ is straightforward: compute the pushout, yielding $Spec_{n+1}$, then compose arrows down the right side of the ladder and the pushout square to obtain $Spec_0 \Rightarrow Spec_{n+1}$ as the final constructed refinement.

Again, rung construction is not simply a matter of computing a colimit. For example,
Figure 13: Taxonomy of Algorithm Theories
there are at least two distinct arrows from *Divide-and-Conquer* to *Sorting*, corresponding to a mergesort and a quicksort – these are distinct cocones and there is no universal sorting algorithm corresponding to the colimit. However, applying the refinement that we select at a node in the taxonomy is a simple matter of computing the pushout. For algorithm design the pushout simply instantiates some definition schemes and other axiom schemes.

It is unlikely that a general automated method exists for constructing rungs of the ladder, since it is here that creative decisions can be made. For general-purpose design it seems that users must be involved in guiding the rung construction process. However in domain-specific settings and under certain conditions it will possible to automate rung construction (as discussed in the next section). Our goal in Designware is to build an interface providing the user with various general automated operations and libraries of standard components. The user applies various operators with the goal of filling out partial morphisms and specifications until the rung is complete. After each user-directed operation, constraint propagation rules are automatically invoked to perform sound extensions to the partial morphisms and specifications in the rung diagram. Constructive theorem-proving provides the basis for several important techniques for constructing classification arrows [7, 9].

### 5.2. Tactics

The design process described so far uses primitive operations such as (1) selecting a spec or refinement from a library, (2) computing the pushout/colimit of (a diagram of) diagram morphisms, and (3) unskolemizing and translating a formula along a morphism, (4) witness-finding to derive symbol translations during the construction of classification arrows, and so on. These and other operations can be made accessible through a GUI, but inevitably, users will notice certain patterns of such operations arising, and will wish to have macros or parameterized procedures for them, which we call *tactics*. They provide higher level (semiautomatic) operations for the user.

The need for at least two kinds of tactics can be discerned.

1. **Classification tactics** control operations for constructing classification arrows. The divide-and-conquer theory admits at least two common tactics for constructing a classification arrow which we illustrated in Section 3.3.1. One tactic can be procedurally described as follows: (1) the user selects an operator symbol with a DRO requirement spec, (2) the system analyzes the spec to obtain the translations of the DRO symbols, (3) the user is prompted to supply a standard set of constructors on the input domain $D$, (4) the tactic performs unskolemization on the composition relation in each Soundness axiom to derive a translations for $O_Ci$, and so on. This tactic was followed in the mergesort derivation.

The other tactic is similar except that the tactic selects constructors for the composition relations on $R$ (versus $D$) in step (3), and then uses unskolemization to solve for decomposition relations in step (4). This tactic was followed in the quicksort derivation.
A classification tactic for context-dependent simplification provides another example. Procedurally: (1) user selects an expression $expr$ to simplify, (2) type analysis is used to infer translations for the input and output sorts of $expr$, (3) a context analysis routine is called to obtain contextual properties of $expr$ (yielding the translation for $C$), (4) unskolemization and witness-finding are used to derive a translation for $new-expr$.

2. **Refinement tactics** control the application of a collection of refinements; they may compose a common sequence of refinements into a larger refinement step. Planware has a code-generation tactic for automatically applying spec-to-code interlogic morphisms. Another example is a refinement tactic for context-dependent simplification; procedurally, (1) use the classification tactic to construct the classification arrow, (2) compute the pushout, (3) apply a substitution operation on the spec to replace $expr$ with its simplified form and to create an isomorphism. Finite Differenting requires a more complex tactic that applies the tactic for context-dependent simplification repeatedly in order to make incremental the expressions set up by applying the $Expression-and-Function \rightarrow Abstracted-Op$ refinement.

We can also envision the possibility of metatactics that can construct tactics for a given class of tasks. For example, given an algorithm theory, there may be ways to analyze the sorts, ops and axioms to determine various orders in constructing the translations of classification arrows. The two tactics for divide-and-conquer mentioned in Section 3.3.1. are an example.

6. **Summary**

Perhaps the main message of this paper is that a formal software refinement process can be supported by automated tools, and in particular that libraries of design knowledge can be brought to bear in constructing refinements for a given requirement specification. One goal of this paper has been to show that diagram morphisms are adequate to capture design knowledge about algorithms, data structures, and expression optimization techniques, as well as the refinement process itself. We showed how to apply a library refinement to a requirement specification by constructing a classification arrow and computing the pushout. We discussed how a library of refinements can be organized into taxonomies and presented techniques for constructing classification arrows incrementally. The examples and most concepts described are working in the Specware, Designware, and Planware systems.

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References


A Containers, Bags, and Sequences

In this section we present parts of the following taxonomy of Container structures and their refinements. This taxonomy is adapted from the Specware/Designware library. One refinement is given in detail: Bag \(\Rightarrow\) Seq.

\[
\begin{array}{c}
\text{Container} \\
\downarrow \\
\text{Proto-Seq} \\
\downarrow \\
\text{Proto-Bag} \\
\downarrow \\
\text{Bag} \\
\downarrow \\
\text{Proto-Set} \\
\end{array}
\]

spec CONTAINER is
  sorts E, Container
  op empty : -> Container
  op singleton : E -> Container
  op _join_ : Container, Container -> Container
  axiom unit is
    x join empty = x & empty join x = x
  constructors {empty, singleton, join} construct Container

  op empty? : Container -> Boolean
  definition of empty? is
    axiom empty?(empty) = true
    axiom empty?(singleton(y)) = false
    axiom empty?(U join V) = (empty?(U) & empty?(V))
  end-definition

  op nonempty? : Container -> Boolean
  def nonempty?(b) = ~empty?(b)

  op _in_ : E, Container -> Boolean
  definition of in is
    axiom x in empty = false
    axiom x in singleton(y) = (x = y)
    axiom x in U join V = (x in U or x in V)
  end-definition
end-spec
spec PROTO-SEQ is
translate
colimit of
diagram
    nodes ASSOCIATIVE, CONTAINER, BIN-OP
    arcs BIN-OP -> ASSOCIATIVE : {}
        , BIN-OP -> CONTAINER : {E -> Container, binop -> join}
end-diagram
by {Container -> Seq,
    empty -> empty-seq,
    empty? -> empty-seq?,
    singleton -> singleton-seq,
    join -> concat}

The definition of PROTO-SEQ given above evaluates to the following “flat” spec:

spec PROTO-SEQUENCE is
    sorts E, Seq
    const empty-seq : -> Seq
    op singleton-seq : E -> Seq
    op _concat_ : Seq, Seq -> Seq
    constructors {empty-seq, singleton-seq, concat} construct Seq

    axiom unit is
        forall(x:Seq)(x concat empty-seq = x & empty-seq concat x = x)
    axiom associativity is
        x concat (y concat z) = (x concat y) concat z

    op empty-seq? : Seq -> Boolean
    definition of empty-seq? is
        axiom empty-seq?(empty) = true
        axiom empty-seq?(singleton(y)) = false
        axiom empty-seq?(U concat V) = (empty-seq?(U) & empty-seq?(V))
end-definition

    op nonempty-seq? : Seq -> Boolean
    def nonempty-seq?(b) = ~empty-seq?(b)

    op _in_ : E, Seq -> Boolean
    definition of in is
        axiom x in empty-seq = false
        axiom x in singleton-seq(y) = (x = y)
        axiom x in U concat V = (x in U or x in V)
end-definition
end-spec
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spec PROTO-BAG is
  translate
colimit of
diagram
  nodes BIN-OP, COMMUTATIVE, PROTO-SEQ
  arcs  BIN-OP -> COMMUTATIVE : {}
        , BIN-OP -> PROTO-SEQ : {E -> Seq, binop -> concat}
end-diagram
by {Seq -> Bag,
    empty-seq -> empty-bag,
    empty-seq? -> empty-bag?,
    singleton-seq -> singleton-bag,
    concat -> bag-union}

spec PROTO-SET is
  translate
colimit of
diagram
  nodes BIN-OP, IDEMPOTENT, PROTO-BAG
  arcs  BIN-OP -> IDEMPOTENT : {}
        , BIN-OP -> PROTO-BAG : {E -> Bag, binop -> bag-union}
end-diagram
by {Bag -> Set,
    empty-bag -> empty-set,
    singleton-bag -> singleton-set,
    bag-union -> union}
We must extend the PROTO-specs with additional sorts and operations to get useful
theories. In practice there would be several extensions to Proto-SEQ yielding a subtax-
onomy of Sequence theories, each useful in different contexts.

Note the use of subsort definitions here. Sub sorts are necessitated by the requirement
that all operators be total. For example, NE-seq (short for Non-Empty sequences) is
defined as sequences restricted to nonempty sequences. The selectors head and tail can
then be introduced as with signature NE-seq \rightarrow Seq. A subsort construction \( E|p? \) comes
with an injective coercion function \( relax(p?) : E|p? \rightarrow E \) that takes subsort elements into
the parent sort. The function \( r \) is introduced as an abbreviation of \( relax(nonempty-seq?) \)
which coerces from the subsort (NE-seq) to the parent sort (seq). Note also how SEQ
introduces destructors as inverses of constructors. These play a key role in divide-and-
conquer-style algorithms. \textit{Unit} denotes the empty product construction – the sort with
one element, \( \langle \rangle \).

spec SEQ is
import PROTO-SEQ

sort NE-Seq
sort-axiom NE-Seq = Seq | nonempty-seq?
op r: NE-seq \rightarrow Seq
def r = relax(nonempty-seq?)

op prepend : E , Seq \rightarrow Seq
op head : NE-Seq \rightarrow E
op tail : NE-Seq \rightarrow Seq
constructors \{empty-seq, prepend\} construct Seq
axiom prepend(a,S) = singleton(a) concat S

op size: Seq \rightarrow Nat
definition of size is
axiom size(empty-seq) = 0
axiom size(prepend(a,S)) = 1 + size(S)
end-definition

sort 0-Seq
sort-axiom 0-Seq = Seq | empty-seq?
op r0: 0-seq \rightarrow Seq
def r0 = relax(empty-seq?)

op empty-seq-inv : 0-seq \rightarrow Unit
inversion of constructor empty-seq is

\text{discriminator empty-seq?}
\text{destructors} \: [\text{empty-seq-inv}]
axiom fa(x:0-seq) empty-seq(empty-seq-inv(x)) = r0(x)

sort 1-Seq
sort-axiom 1-Seq = Seq | singleton-seq?

op singleton-seq? : Seq -> Boolean

def singleton-seq?(x) = (size(x)=1)

op r1: 1-seq -> Seq

def r1 = relax(singleton-seq?)

op singleton-seq-inv : 1-Seq -> E

inversion of constructor singleton-seq is

discriminator singleton-seq?

destructors [singleton-seq-inv]

axiom fa(x:1-Seq) singleton-seq(singleton-seq-inv(x)) = r1(x)

sort 2-Seq

sort-axiom 2-Seq = Seq | nonsingleton-seq?

op nonsingleton-seq? : Seq -> Boolean

def nonsingleton-seq?(x) = (size(x)>1)

op r2: 2-seq -> Seq

def r2 = relax(nonsingleton-seq?)

op left-split : 2-Seq -> Seq

op right-split : 2-Seq -> Seq

inversion of constructor concat is

discriminator nonsingleton-seq?

destructors [left-split, right-split]

axiom fa(x:2-Seq) concat(left-split(x), right-split(x)) = r2(x)

end-spec

spec BAG is

import PROTO-BAG

op size: Bag -> Nat

definition of size is

axiom size(empty-bag) = 0

axiom size(singleton-bag(e)) = 1

axiom size(bag-union(b1, b2)) = size(b1) + size(b2)

end-definition

op bag-wflgt : Bag, Bag -> Boolean

def bag-wflgt(b1, b2) = (size(b1) > size(b2))

op singleton-bag? : Bag -> Boolean

def singleton-bag?(b) = (size(b)=1)

op nonsingleton-bag? : Bag -> Boolean

def nonsingleton-bag?(b) = (size(b)>1)

end-spec

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The following text gives the refinement of bags to sequences, as described in Section 2.4.3. Note the use of a quotient sort definition here. The sort Bag-as-Seq is defined as sequences quotiented by the permutation relation. The purpose of Bag-as-Seq is to serve as the translation of Bag. The quotienting captures the commutativity of bags: we say two sequences are equal as bags exactly when they are permutations of one another. When we finally translate down to code, equality on bags will translate to the permutation relation on the implementing sequences. A quotient sort construction $E/\equiv$ comes with a surjective coercion function $\text{quotient}(\equiv): E \to E/\equiv$ that takes $E$ elements to their equivalence class. The function $q$ is introduced as an abbreviation of $\text{quotient}(\text{perm})$.

spec BAG-AS-SEQ is
import SEQ

op remove-1 : E, Seq -> Seq
definition of remove-1 is
  axiom empty-seq?(S) => remove-1(x S)=S
  axiom x=head(S) => remove-1(x, r(S))=tail(S)
  axiom ~(x=head(S))
    => remove-1(x, r(S))=prepend(head(S), remove-1(x, tail(S)))
end-definition

op perm? : Seq, Seq -> Boolean
definition of perm? is
  axiom empty-seq?(S1)=> (perm?(S1 S2) = empty-seq?(S2))
  axiom perm?(r(S1), S2)
    => head(S1) in S2 & perm?(tail(S1), remove-1(head(S1), S2))
end-definition

sort Bag-as-Seq
sort-axiom Bag-as-Seq = Seq/perm?
constructors {bag-empty, bag-singleton, bag-union} construct Bag-as-Seq

op q : Seq -> Bag-as-Seq
def q = quotient(perm?)

op bag-empty : Bag-as-Seq
def bag-empty = q(empty-seq)

op bag-empty? : Bag-as-Seq -> Boolean
def bag-empty?(q(s)) = empty-seq?(s)

op bag-nonempty? : Bag-as-Seq -> Boolean
def bag-nonempty?(q(s)) = ~empty-seq?(s)

op bag-singleton : E -> Bag-as-Seq
def bag-singleton(x) = q(singleton-seq(x))

op bag-singleton? : Bag-as-Seq -> Boolean
def bag-singleton?(q(s)) = singleton-seq?(s)

op bag-nonsingleton? : E -> Bag-as-Seq
def bag-nonsingleton(q(s)) = nonsingleton-seq?(s)

op bag-in : E, Bag-as-Seq -> Boolean
def bag-in(x, q(S)) = x in S

op bag-union : Bag-as-Seq, Bag-as-Seq -> Bag-as-Seq
def bag-union(q(S1),q(S2)) = q(concat(S1, S2))

op bag-size : Bag-as-Seq -> Nat
definition of bag-size is
   axiom bag-size(q(s)) = size(s)
end-definition

op bag-wflgt : Bag-as-Seq, Bag-as-Seq -> Boolean
definition of bag-wflgt is
   axiom bag-wflgt(b1, b2) = gt(bag-size(b1), bag-size(b2))
end-definition

end-spec

Finally the refinement from BAG to SEQ-AS-BAG is

diagram-morphism BtoS is
   {BtoS-Triv : Triv -> Triv,
    BtoS-Bag : Bag -> Seq-as-Bag}

diagram BAG is
   nodes Bag, Triv
   arcs Triv -> Bag: {E-> E}
end-diagram

diagram BAG-AS-SEQ is
   nodes Seq, Bag-As-Seq, Triv
   arcs Triv -> Bag-As-Seq: {E-> E},
      Triv -> Seq : {E -> E},
      Seq -> Bag-As-Seq: import-morphism
end-diagram

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morphism BtoS-Bag : Bag -> Bag-As-Seq is
{Bag -> Bag-As-Seq,
empty-bag -> bag-empty,
empty-bag? -> bag-empty?,
nonempty? -> bag-nonempty?,
singleton-bag -> bag-singleton,
singleton-bag? -> bag-singleton?,
nonsingleton-bag? -> bag-nonsingleton?,
in -> bag-in,
bag-union -> bag-union,
bag-wfgt -> bag-wfgt,
size -> bag-size}
B Specification for Sorting

Listed below are the components of the structured specification for the problem of sorting a bag whose elements are drawn from a linearly ordered set. The specification is parameterized on the linear order.

```
spec BAG-AND-SEQ is
  colimit of diagram
  nodes TRIV, BAG, SEQ
  arcs TRIV -> BAG: {},
       TRIV -> SEQ: {}
end-diagram

spec BAG-AND-SEQ-CONV is
  import BAG-AND-SEQ
  op seq-to-bag: Seq -> Bag
  definition of seq-to-bag is
    axiom seq-to-bag(empty-seq) = empty-bag
    axiom seq-to-bag(singleton-seq(e)) = singleton-bag(e)
    axiom seq-to-bag(concat(S1, S2))
        = bag-union(seq-to-bag(S1), seq-to-bag(S2))
  end-definition
end-spec

spec BAG-SEQ-LinOrd is
  colimit of diagram
  nodes TRIV, LINEAR-ORDER, BAG-AND-SEQ-CONV
  arcs TRIV -> LINEAR-ORDER: {},
```
spec BAG-SEQ-over-LinOrd is
  import BAG-SEQ-LinOrd

  op all-le: Bag, Bag -> Boolean
  definition of all-le is
  axiom all-le(empty-bag, empty-bag) = true
  axiom all-le(B, empty-bag) = true
  axiom all-le(singleton-bag(e), empty-bag) = true
  axiom all-le(singleton-bag(d), singleton-bag(e)) = (d le e)
  axiom all-le(singleton-bag(d), bag-union(B1, B2))
    = all-le(singleton-bag(d), B1) & all-le(singleton-bag(d), B2)
  axiom all-le(bag-union(B1, B2), B) = all-le(B1, B) & all-le(B2, B)
  end-definition

  op ordered?: Seq -> Boolean
  definition of ordered? is
  axiom ordered?(empty-seq) = true
  axiom ordered?(singleton(a)) = true
  axiom ordered?(prepend(a, y)) = all-le(singleton(a), seq-to-bag(y))
  axiom ordered?(concat(y, z)) = ordered?(y)
    & all-le(seq-to-bag(y), seq-to-bag(z)) & ordered?(z)
  end-definition
end-spec

spec SORTING is
  import BAG-SEQ-over-LinOrd

  op sorted?: Bag, Seq -> Boolean
  def sorted?(x, z) = (ordered?(z) & x = seq-to-bag(z))

  op sorting: Bag -> Seq
  axiom sorted?(x, sorting(x))
end-spec