Big Omega Versus the Wild Functions*

Paul M.B. Vitányi
Lambert Meertens

Centre for Mathematics & Computer Science (C.W.I.), Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

ABSTRACT

The question of the desirable properties and proper definitions of the Order-of-Magnitude symbols, in particular O and Θ, is addressed once more. The definitions proposed are chosen for complementary mathematical properties, rather than for similarity of form.

> The old order changeth, yielding place to new,
> And God fulfils himself in many ways,
> Lest one good custom should corrupt the world.
>
> Tennyson, *The Idylls of the King.*

1. Introduction

The issue of the proper definitions for the Order-of-Magnitude symbols would appear to have been settled once and for all by Knuth in [1]. At the end of an exhaustive discussion the subject is, the author feels, about "beaten to death". The purpose of this communication is to point out that there is life in the old dog yet.† The deliberations below were prompted by surprise that, while proving a lower bound where the precise definitions mattered, matters were not as clear-cut as one might assume them to be. When we want to prove something about the order of magnitude of a function we do not know, like the worst-case running time of some algorithm, we can not assume that the function concerned does not oscillate or — as can be the case for monotonic functions — that the limit of the quotient of that function and the measuring function exists at all. In such cases the order of magnitude of a function may vary arbitrarily, depending on the precise definitions chosen. This encourages improper use, in particular of the symbol Ω. A modification of the

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† Charles Kingsley, *Two Years Ago.* 1857. "I feel his head, there is life in the old dog yet".

proposal in [1], for the definitions of the Order-of-Magnitude symbols, appears to give a more useful and manageable system.

2. History

The history of the Order-of-Magnitude symbols O, Ω, Θ, o and ω, is explored in [1]. Some additional sources are as follows. In the classic textbook on Analysis by Whittaker & Watson [2] the origin of the founding father O is given as: "This notation is due to Bachmann, *Zahlentheory* (1894), p. 401, and Landau, *Primzahlen, I*, (1909), p. 61". The *Encyklopädie der Mathematischen Wissenschaften* contains, not surprisingly, occurrences of the Order-of-Magnitude symbol O in a section on Analytical Number Theory by Bachmann himself [3, p. 664], and also the equivalent, more ancient, relational symbols $\prec$, $\sim$ and $\succ$. These symbols correspond, more or less, to the symbols o, Θ, and ω, respectively; cf. [3, p. 75]. They are attributed to Du Bois-Reymond [4], and are said to hold between two functions, *only if* the limit of the quotient of the functions exists. So $f(n) \sim g(n)2$ if $\lim f(n)/g(n) \rightarrow c2$, $0 < c < \infty 2$, for $n \rightarrow \infty 2$. The author of this section of [3], Pringsheim, states that he uses the notation "$\sim$" also when the lim sup and lim inf of the quotient are distinct but still *both* finite and non-vanishing. So $f(n) \sim g(n)$ if

$$0 < c \leqslant \liminf_{n \rightarrow \infty} \frac{f(n)}{g(n)} \leqslant \limsup_{n \rightarrow \infty} \frac{f(n)}{g(n)} \leqslant C < \infty \ .$$

He adds the symbol $\cong$ for

$$f(n) \cong c\, g(n), \text{ if } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c \ .$$

Hardy used $\sim$ in the (now standard) sense of Pringsheim's $\cong$, or Du Bois-Reymond's $\sim$ with $c = 1$, and the sign $\asymp$ for Du Bois-Reymond's $\sim$ in Pringsheim's interpretation. To complete the descendancy of the Order-of-Magnitude symbols as told in [1]: Ω is introduced in [9, p. 225], Θ on suggestions of R. E. Tarjan and M. S. Paterson in [1, p. 20], where the ω notation also appears first.

The potential inequality of the lim sup and the lim inf of the quotient of two functions, even if both are monotonic, cf. [5], is both a source of discord and of the present note. In questions of classical Analysis this inequality was a troublesome matter. Therefore Hardy [6,7,8] constructed a well-behaved class of "tame" functions, the L-functions, which made it possible to formulate the problems concerned precisely enough and to reason rigorously. A similar class of functions was studied earlier by Liouville. Hardy [7] considers an L-function

> "essentially as the embodyment of an 'order of infinity', as expressing a certain rate of increase or decrease or of approach to a limit; and for this reason I consider only functions of a real variable which are real and one-valued and (as I shall show) ultimately monotonic, excluding altogether oscillating functions such as $\sin x$. These ideas do not appear in Liouvilles work at all. He was interested solely in problems of functional form: $\sin x$ was for him exactly on the same footing as $\log x$ or e^x."

These L-functions will figure prominently in what follows. The departure of the definitions presented here from the proposal in [1] basically rests on the difference in defining Ω. In

[1] the course is taken to give all Order-of-Magnitude symbols the same definitional form. Yet the choice is also a matter of usefulness in expressing the things we want to express; of how well it fits our mathematical intuition, in particular, whether it has elegant properties that can be relied on and used to prove statements without meticulous reference to the original definition. We shall strive for complementary mathematical properties. The use of the symbols o, O and Ω in classical mathematics, as in [6,7,8,9,10], is the same as ours (the H-variant in Section 5). The corresponding definitions for Θ and ω follow easily. The Knuth proposals in [1], that is, (o_K), (O_K), (Θ_K), (Ω_K) and (ω_K) in Section 5, are just alternative notations for the established meaning of the well-known symbols $\prec$, $\preccurlyeq$, $\asymp$, $\succcurlyeq$ and $\succ$, respectively.

3. Usefulness

To dot the *i*'s for *very* formal thinkers we detect some abuse of notation.

> *Notational convention.* Some convenient abuse of notation is universally practised when dealing with the Order-of-Magnitude symbols. An expression E depending on a variable n, such as n^2, which usually denotes the *value* of E for a given value of n, is also used to denote the *function* that maps n to E, $\lambda n\,[E]$ (so n^2 may denote the squaring function). The intended use will be clear from the context. This allows us to write $\mathrm{O}(n^2)$ rather than $\mathrm{O}(\lambda n\,[n^2])$.

Suppose we can prove that the running time $T(n)$ of some algorithm exceeds n^2 infinitely often and also that it is never less than $n \log n$. We would like to be able to say then that $T(n) \in \Omega(n^2)$. According to the definitions in [1] the only thing we can say about $T(n)$ is that $T(n) \in \Omega(n \log n)$, which is not very informative. We do not want to express that the algorithm will *always* exceed a (rather weak) time limit; we want to express that it will exceed a time limit for arbitrarily large instances of the data. To refresh the reader's memory, we repeat here the definition for Ω as proposed by Knuth in [1]. Those for the other symbols can be found in Section 5.

(Ω_K) $\Omega(f(n))$ denotes the set of all $g(n)$ such that there exist positive constants δ and n_0 with $g(n) \geqslant \delta f(n)$ for all $n \geqslant n_0$.

In the above example, the problem could be ascribed to our lack of knowledge about the function $T(n)$. But the problem may also arise with functions that are fully known. Consider

$$g(n) = \exp_2(\exp_2 \lfloor \log_2 \log_2 n \rfloor).$$

We have $g(n) \leqslant n$ for all n, and $g(n) = n$ for n of the form 2^{2^t}. Also, $g(n)$ is monotonic non-decreasing. (It is easy to make a variant that is continuous and monotonic increasing.) Here we would like to assert that $g(n) \in \Omega(n)$, but according to (Ω_K), the largest "tame" function $f(n)$ such that $g(n) \in \Omega(f(n))$ is $f(n) = n^{1/2}$. Nonetheless, the least "tame" function $f(n)$ such that $g(n) \in \mathrm{O}(f(n))$ is $f(n) = n$. (The concept of a "tame" function plays a crucial part here. The *logarithmico-exponential* functions or L-functions, introduced by Hardy [6,7,8] to calibrate the orders of magnitude, constitutes an appropriate family of "tame" functions, cf. Sections 4 and 5.) Whereas in giving an upper bound we generally want to express that a function is in some sense confined by that upper bound, in stating a lower bound we want to express non-confinement. The problem would be remedied if in definition (Ω_K) we replace "for all $n >$ [some positive] n_0" by "for infinitely many n". As long as all functions concerned are tame, this will make no difference.

4. Intuition

For the function $g(n)$ introduced above, we have, according to definition (Ω_K),

$$g(n) \in \Omega(n^{1/2}) \cap O(n),$$

but, for all $\epsilon > 0$,

$$g(n) \notin \Omega(n^{1/2+\epsilon}) \cup O(n^{1-\epsilon}).$$

Such consequences from the definitions in [1] seem contrary to intuition. One of the motivations for [1] was to counter the improper use of the symbol O where Θ would (now) be appropriate. Since the appearance of this seminal paper in 1976, the use of the Order-of-Magnitude symbols, both proper and improper, has become far more customary than before. For that very reason, juggling with Order-of-Magnitude symbols along the lines of "By way of contradiction, suppose $f(n)$ is not in $\Omega(n)$. Therefore, $f(n)$ is in $o(n)$, and so ..." has become more attractive. This is so, because the meanings of the Order-of-Magnitude symbols on the tame functions are such that O corresponds with $\leqslant$, Ω corresponds with $\geqslant$, Θ corresponds with $=$, o corresponds with $<$ and ω corresponds with $>$. Since, in practice, almost all functions are tame enough, one tends to forget that this correspondence does not extend to all functions. However, in improper reasoning such as that above, the function $f(n)$ under consideration may be unknown, and not known to be tame. Rather than to rebuke authors who indulge in such practice (while referring to [1] for definitions), we would like to see a definition of the Order-of-Magnitude symbols that legitimizes this kind of reasoning. Let the function $f(n)$ be fixed, so that we can simply write O for $O(f(n))$, and similarly for the other Order-of-Magnitude symbols. The meanings of o and O are well established in mathematical practice and will not be disputed here. However, note that under the standard definition both o and O may contain negative functions, such as $-n$. For the purpose of the discussion, it is convenient if we can restrict our attention to the set of "non-negative functions", where a function is non-negative if it assumes no negative values for sufficiently large values of n. Denote this set by U. If $f(n)$ is a non-negative function, then each of the sets Θ, Ω and ω is contained in U under any reasonable definition, including that in [1]. So let us write (only here) o while meaning $o \cap U$, and similarly for O. The properties we want to have now are:

(Ω_H) $\quad \Omega = U - o,$

(ω_H) $\quad \omega = U - O,$

and

(Θ_H) $\quad \Theta = O \cap \Omega.$

We can use these desirable properties as definitions. If one uses Knuth's definitions, the first two of these three properties are not assured in general (but do hold if restricted to tame functions). The meaning of Ω according to definition (Ω_H) is the same as the one proposed by way of remedy at the end of the previous section. It is a consequence of the new definitions that

$$\mathrm{O} = o \cup \Theta \text{ and } \Omega = \Theta \cup \omega.$$

The Order-of-Magnitude symbols o, O, Θ, Ω and ω now have by and large the same general properties as the usual $<$, $\leqslant$, $=$, $\geqslant$ and $>$. For example, just as we conclude $x<y$ from $x \leqslant y$ and $x \neq y$, we may conclude that $g(n) \in o(f(n))$ from $g(n) \in \mathrm{O}(f(n))$ and $g(n) \notin \Theta(f(n))$. However, we may (analogously to $y>x$ following from $x<y$), conclude $f(n) \in \omega(g(n))$ from $g(n) \in o(f(n))$, but not *vice versa*. Similarly, if $g(n) \in \Theta(f(n))2$, we may not, in general, conclude that $f(n) \in \Theta(g(n))$. Thus, we have lost a pleasant property, since these very conclusions were valid under the definitions in [1]. However, we feel that the gain is worth the loss; in the practice of reasoning with these symbols, such a switch of roles between the measured and the measuring function is rare. The stronger relation $g(n) \in \Theta(f(n))$, in the sense of Knuth, may still be expressed, viz. as $\Theta(g(n)) = \Theta(f(n))$.

5. Formal analysis of proposals

The definitions proposed in [1] look as follows.

(o_K) $\quad o(f(n)) = \{g \mid \forall_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[\,|g(n)| < \delta f(n)]\,\};$

(O_K) $\quad \mathrm{O}(f(n)) = \{g \mid \exists_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[\,|g(n)| \leqslant \delta f(n)]\,\};$

(Θ_K) $\quad \Theta(f(n)) = \{g \mid \exists_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[g(n) \leqslant \delta f(n)]$
$\quad\quad \& \ \exists_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[g(n) \geqslant \delta f(n)]\,\};$

(Ω_K) $\quad \Omega(f(n)) = \{g \mid \exists_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[g(n) \geqslant \delta f(n)]\,\};$

(ω_K) $\quad \omega(f(n)) = \{g \mid \forall_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[g(n) > \delta f(n)]\,\}.$

The definitions proposed here are:

(o_H) $\quad o(f(n)) = \{g \mid \forall_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[\,|g(n)| < \delta f(n)]\,\};$

(O_H) $\quad \mathrm{O}(f(n)) = \{g \mid \exists_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[\,|g(n)| \leqslant \delta f(n)]\,\};$

(Θ_H) $\quad \Theta(f(n)) = \{g \mid \exists_{\delta>0}\exists_{n_0>0}\forall_{n>n_0}[g(n) \leqslant \delta f(n)]$
$\quad\quad \& \ \exists_{\delta>0}\forall_{n_0>0}\exists_{n>n_0}[g(n) \geqslant \delta f(n)]\,\};$

(Ω_H) $\quad \Omega(f(n)) = \{g \mid \exists_{\delta>0}\forall_{n_0>0}\exists_{n>n_0}[g(n) \geqslant \delta f(n)]\,\};$

(ω_H) $\quad \omega(f(n)) = \{g \mid \forall_{\delta>0}\forall_{n_0>0}\exists_{n>n_0}[g(n) > \delta f(n)]\,\}.$

We shall compare the two proposals using the set of calibrating L-functions defined in [6]. *Definition.* The L-functions are the smallest class of real one-valued functions of a real variable n, containing the constant functions and n, and closed under the arithmetic operations, exp and log. The requirement of an L-function being real one-valued is satisfied if it is so for all values of n greater than some n_0.

The fundamental theorem on L-functions in [6,7,8] then is as follows.

Theorem. *Any L-function is ultimately continuous, of constant sign, and monotonic; and tends, as* $n \to \infty$, *to* ∞, *or to zero or to some other definite limit. Further, if* f *and* g *are L-functions, one or other of the relations* $f \in o(g)$, $f \in \Theta(g)$ *or* $f \in \omega(g)$ *holds between them.*

As Hardy remarks, if f and g are L-functions then f/g is an L-function. Thus, the second part of the theorem is a mere corollary of the first part; for it follows that f/g must tend to infinity, or to zero or some other limit. In the family of L-functions, therefore, the K-or-H choices of definitions for the Order-of-Magnitude symbols do not matter; the theorem is insensitive to the variations of definitions above. The theorem ensures that the L-functions are suitable for the purpose of calibrating the order of increase of functions, since they are totally ordered by the Order-of-Magnitude symbols. That is, o, O, Θ, Ω and ω have precisely the same roles on the set L-functions as $<$, $\leqslant$, $=$, $\geqslant$ and $>$ have on the rationals.

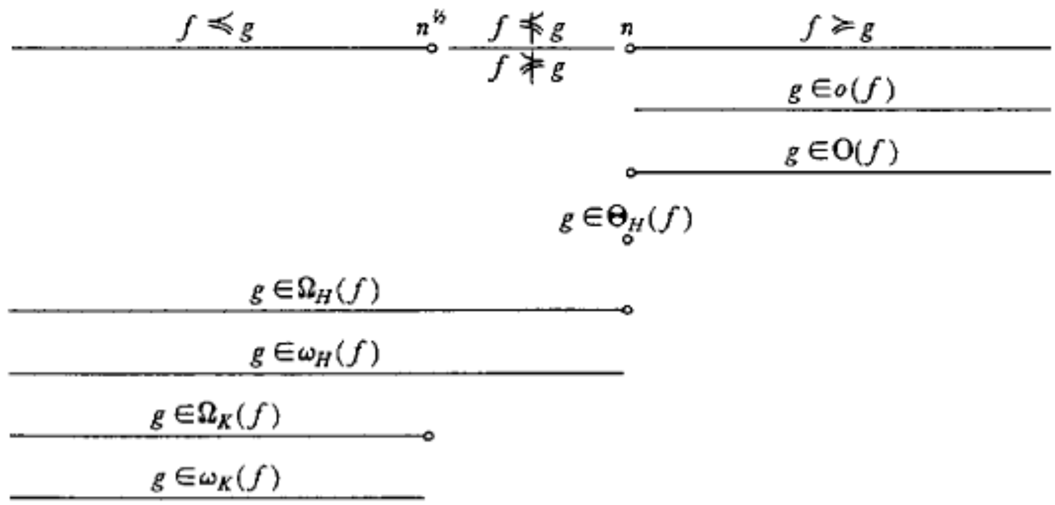

Figure. The top line depicts the ordered set of L-functions. The subsets of the set of L-functions induced by the different Order-of-Magnitude symbols, with respect to the wild function $g(n) = \exp_2(\exp_2\lfloor \log_2 \log_2 n \rfloor)$, correspond to the labelled line segments. N.B., we have $\forall_{f \in L} [g \notin \Theta_K(f)]$.

In the Figure we sketch the meaning of the several proposals, couched in terms of the tame L-functions f, for a given wild function g (here $g(n) = \exp_2(\exp_2\lfloor \log_2 \log_2 n \rfloor)$). We subscript the symbols with H or K, to distinguish the different proposals referred to, whenever the intended meaning is not clear from the context. Thus, $f \in \Omega_K(g)$ iff $f \succcurlyeq g$, and so forth. For a tame function g the wild middle gap (where $f \not\preccurlyeq g$ and $f \not\succcurlyeq g$) shrinks to zero and the K- and H-definitions coincide. The wilder a function, as compared to a tame class like the L-functions, the more the two defining methods will differ. The reader should construct such a picture for the function $g_1(n) = \exp(n \sin n)$, or, more difficult, for a function g_2 such that $g_2(g_2(n)) = \exp(n)$. Note that the latter is wild in another sense than the former. While g and g_1 cover a segment of the ordered set of L-functions, by being of irregular increase, g_2 falls in a gap in the ordered set of L-functions; that is, although the increase of the function does not oscillate from that of one L-function to that of another, there is no L-function capable of measuring it [6]. The question presents itself, whether the family of L-functions is rich enough a class to calibrate the functions we meet and wish to calibrate. According to Hardy [6, p. 32]

> "... it is possible, in a variety of ways, to construct functions whose increase cannot be measured by any L-function. It is none the less true that no one has yet succeeded in defining a mode of increase which is genuinely independent of all logarithmico-exponential modes. ... No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms. It would be natural to expect that the arithmetical functions which occur in the theory of numbers might give rise to genuinely new modes of increase; but, so far as analysis has gone, the evidence is the other way."

Thus, *really* wild functions appear to have been a rare species. Seldom seen in the wild, they had to be cultured under laboratory conditions. This state of affairs may be unchanged in the realm of mathematical analysis. However, in computer science the area of algorithms and computational complexity has enriched the taxonomy of "orders of infinity" with a non-L-functions like $\log^*$, the "functional inverse" of Ackermann's function. Note that $\log^*$ is tame enough, but that its rate of increase is far slower than is expressible by unbounded L-functions.

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